Presentations of Premonoidal Categories by Devices

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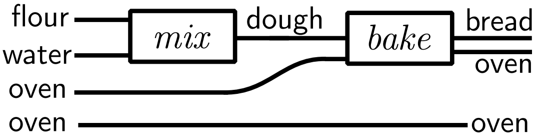
22nd November 2023

String diagrams for process theories

This talk is about a graphical syntax for *processes*, broadly construed.

Monoidal categories are an algebraic formalism for *resource-transforming processes*.

String diagrams are a *sound* and *complete* graphical syntax for monoidal categories.[a]

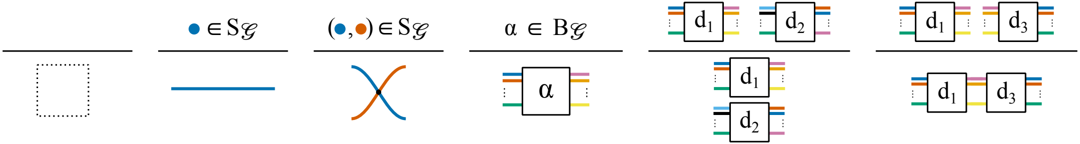


Examples: sets and partial maps with cartesian product, Hilbert spaces and bounded linear maps with tensor product.

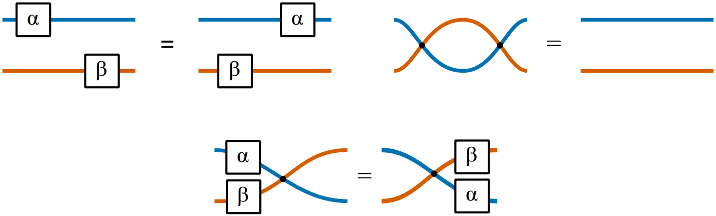
[a] Joyal & Street [JS91]

String diagrams for process theories

Completeness – the free symmetric monoidal category on a set of generating processes is given by string diagrams:



Soundness – equational reasoning is topological:



Premonoidal categories for effectful processes

Interchange is not always obeyed by processes in computer science:

Premonoidal categories[a] refine monoidal categories: interchange does not hold globally.

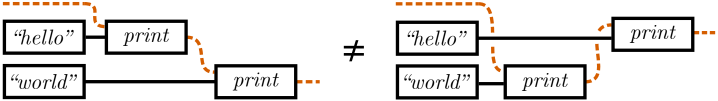
Key example: Kleisli categories of strong monads, or more generally strong promonads.

Interchange holds just when the monad is *commutative*.

[a]Power and Robinson [PR97].

String diagrams for premonoidal categories

Adding a runtime wire presents the free premonoidal category with specified *centre*.[a]



In practice, this global effect limits topological reasoning:

[a] Jeffrey, Román [Jef97, Rom23].

String diagrams with devices

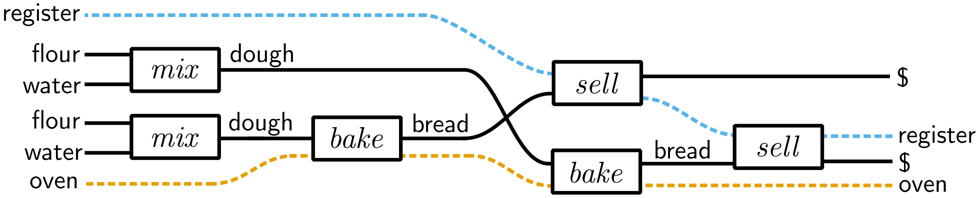
In practice, this global effect limits topological reasoning:

Introduce multiple *device* wires:

Now many natural equations are topological again.

String diagrams with devices

Premonoidal categories are an algebraic foundation for processes that may use both resources and *devices*.



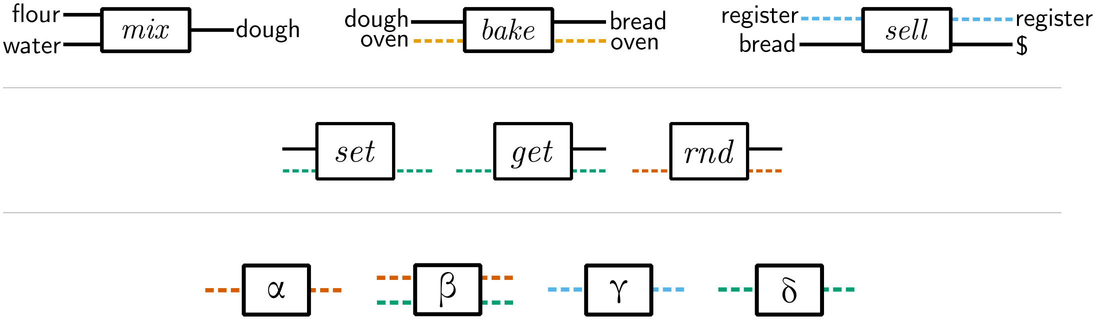
Devices are definite noun phrases: if we only have one oven, we cannot *bake* in parallel.

We introduce a convenient presentation for premonoidal categories based on this idea.

Device signature

Definition. A *device signature* is given by:

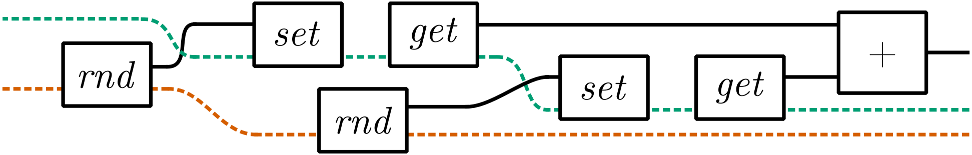
- sets R, P, D of *resources, processes* and *devices*,
- functions $s, t : P \to R^*$ assigning source and target *words* of resources,
- a function $d : P \to \mathscr{P}(D)$ specifying a set of devices used by each process.



Device presentations

A device presentation further specifies some equations between string diagrams:

Proposition. *Device presentations freely generate premonoidal categories.*



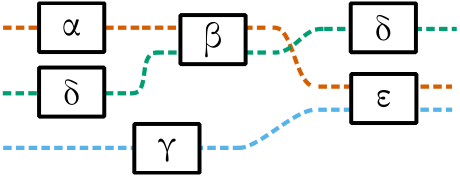
Example adapted from the *functional machine calculus.*[a]

[a]Barrett, Heijltjes & McCusker [Bar23, BHM]

Mazurkiewicz traces by devices

Mazurkiewicz traces [DR95] model the behaviour of concurrent machines.

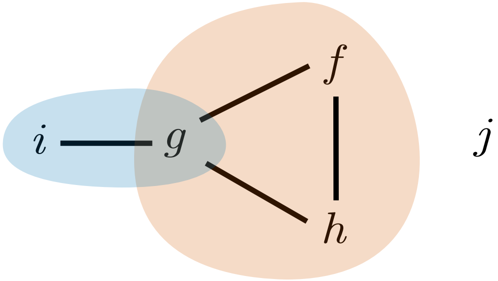
Traces generalize *words*, the behaviour of sequential machines, by allowing specified pairs of actions to commute.

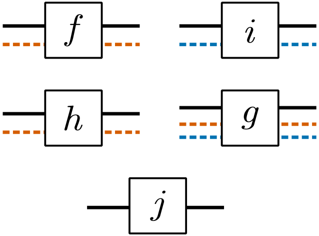


Proposition (E., Sobociński [ES]). *Mazurkiewicz traces arise as the morphisms of premonoidal categories generated by device signatures with no resource wires.*

These devices may be conceived of as shared memory locations.

The canonical device presentation





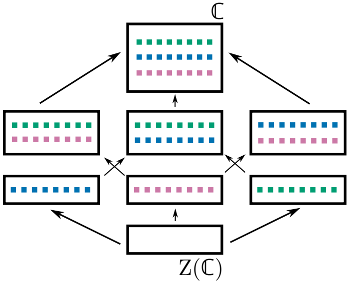
Proposition. *The interference graph of a premonoidal category $\mathbb{C}$ determines a device presentation of $\mathbb{C}$ that contains a device for each non-trivial maximal clique.*

Proposition. *The interference graph of a premonoidal category presented by devices recovers the device presentation.*

The device lattice

The *centralizer* of a set S of processes in a premonoidal category contains all processes interchanging with elements of S.

Proposition. *Centralizers are premonoidal subcategories.*



Proposition. *A premonoidal category $\mathbb{C}$ admits a lattice of premonoidal subcategories, each corresponding to a subset of the devices of $\mathbb{C}$, bounded below by its centre $Z(\mathbb{C})$, and above by $\mathbb{C}$.*

Future directions: Combining effects

Combining categories of effectful processes:

- coproducts and tensor products of algebraic theories [HPP06]
- distributive laws of monads [Bec69]

Given two presentations, we have various ways to combine them.

How do these relate to known constructions for combining effects?

References

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