Monoidal optics are universal

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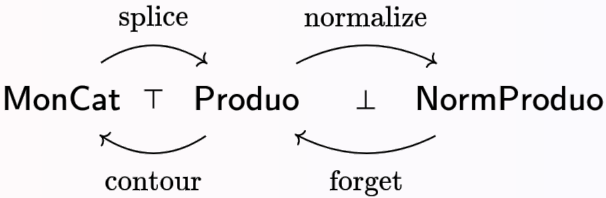
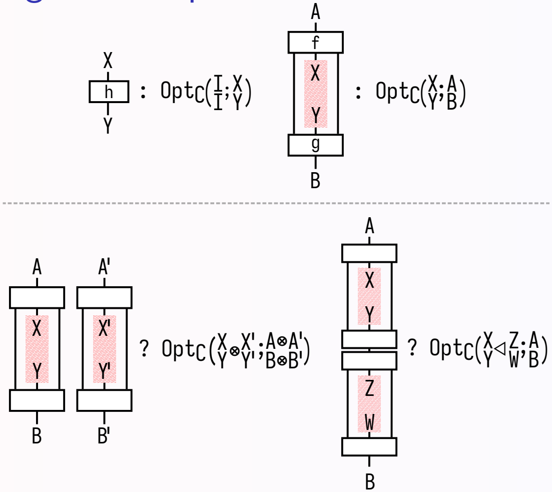
This talk is about the *universal property*
of the *normal produoidal category*
of *optics in a monoidal category*.



E, Hefford, and Román, 2024 – *The Produoidal Algebra of Process Decomposition*

The structure of categories of optics



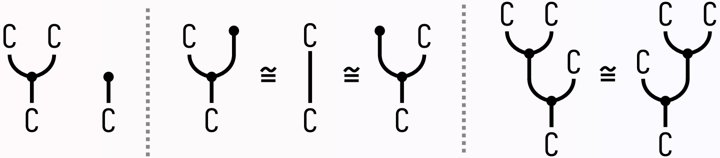
Street, 2012, §4.6 – *Monoidal categories in, and linking, geometry and algebra*

Pastro and Street, 2008 – *Doubles for monoidal categories*

Riley, 2018 – *Categories of optics*

Promonoidal categories

A promonoidal category is a category $\mathcal{C}$ equipped with unit and tensor profunctors



such that these isomorphisms are coherent (pentagon + triangle axioms).

More concretely, we have a category equipped with

- a set $\otimes_{\mathcal{C}}(A, B; C) =: \mathcal{C}(A \otimes B; C)$ for every triple of objects ("splits")
- a set $I(; A) =: \mathcal{C}(I; A)$ for every object ("atoms")
- functorial actions of the morphisms of $\mathcal{C}$ / $\mathcal{C} \times \mathcal{C}$ on these sets
- such that splits are coherent

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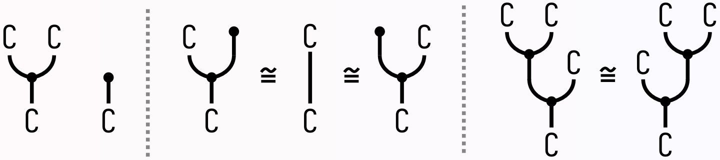
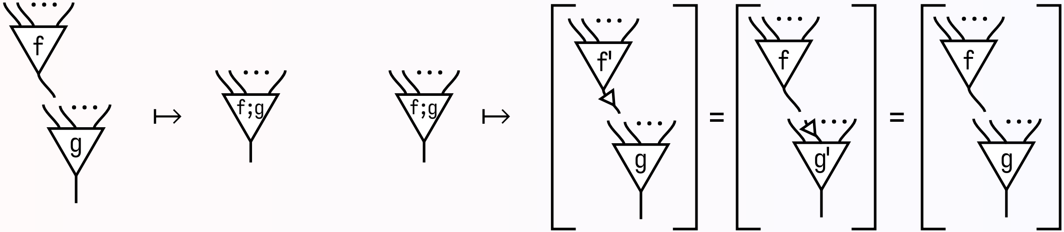


such that these isomorphisms are coherent (pentagon + triangle axioms).

- Every monoidal category is promonoidal
- Closed monoidal structures on presheaves $\simeq$ promonoidal structures on the base (Day, 1970)
- Promonoidal categories $\simeq$ malleable multicategories (M. Román, 2023)
- Optics in a planar monoidal category (Pastro and Street, 2008)
- *Spliced arrows in a category*

Promonoidal categories $\simeq$ malleable multicategories

A multicategory $\mathcal{M}$ is *malleable* if its composition is invertible up to dinaturality.



Theorem M. Román, 2023 – *Monoidal Context Theory*.
The category of promonoidal categories is equivalent to the category of malleable multicategories.

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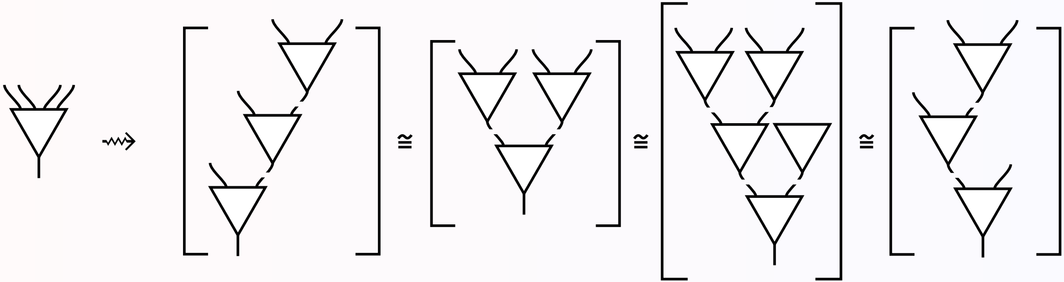
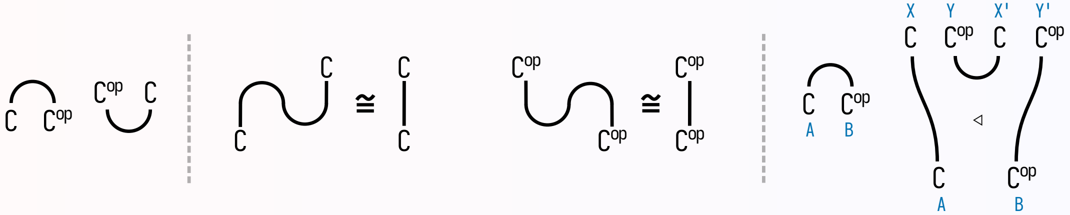


Theorem M. Román, 2023 – *Monoidal Context Theory*.
The category of promonoidal categories is equivalent to the category of malleable multicategories.

Splice: the cofree promonoidal category over a category

$\mathcal{C} \times \mathcal{C}^{op}$ has a promonoidal structure:



$$I(;\,{}^{A}_{B}) = \mathcal{C}(A;B)$$

$$\triangleleft({}^{X}_{Y}, {}^{X'}_{Y'}; {}^{A}_{B}) = \mathcal{C}(A;X) \times \mathcal{C}(Y;X') \times \mathcal{C}(Y';B)$$

$$A \longrightarrow X \qquad Y \longrightarrow X' \qquad Y' \longrightarrow B$$

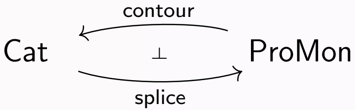
Composing profunctors...

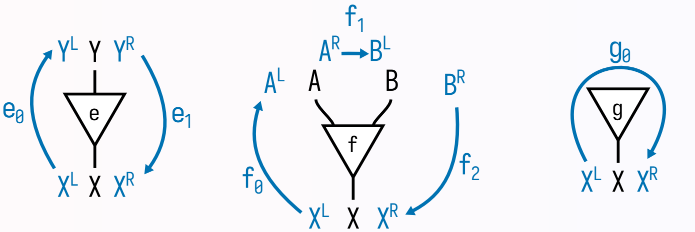
$$\mathcal{C} \times \mathcal{C}^{op} \left({}^{X}_{Y} \triangleleft ({}^{X'}_{Y'} \triangleleft {}^{X''}_{Y''}); {}^{A}_{B}\right) \cong \mathcal{C}(A;X) \times \mathcal{C}(Y;X') \times \mathcal{C}(Y';X'') \times \mathcal{C}(Y'';B)$$

cf. Day, 1970 – *On closed categories of functors* Melliès and Zeilberger, 2023 – *Categorical contours...*

Splicing is right adjoint to contouring

Theorem (E, Hefford, and Román, 2024).

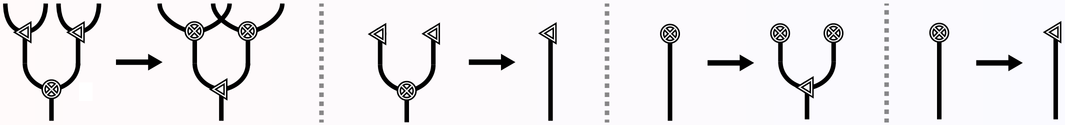




cf. Melliès and Zeilberger, 2023 – *Categorical contours...*

Produoidal categories

A *produoidal category* is a category with two promonoidal structures such that...



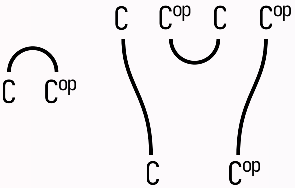
Examples

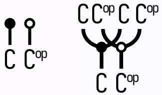
- Every duoidal category is a produoidal category
- Closed duoidal structures on presheaves $\simeq$ produoidal structures on the base
- *Spliced monoidal arrows*
- *Optics in a (planar) monoidal category*

cf. 🗎 Booker and Street, 2013 – *Tannaka Duality and Convolution for Duoidal Categories*

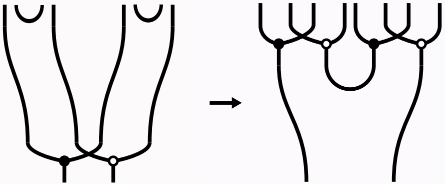
Splice (II): The cofree produoidal category over a monoidal category

When $\mathcal{C}$ is monoidal we have a second promonoidal structure on $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$



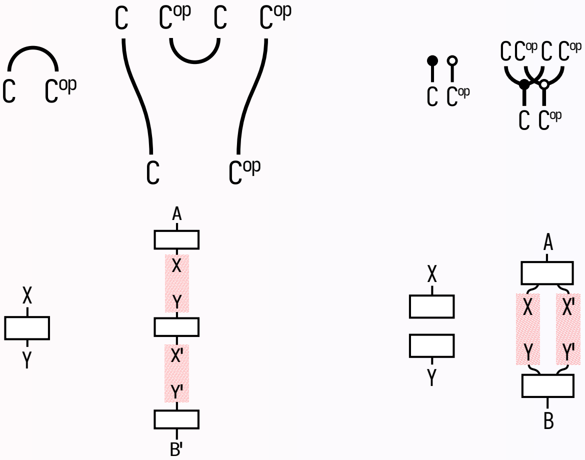


and these interact to form a produoidal structure, e.g.



Splice (II): The cofree produoidal category over a monoidal category

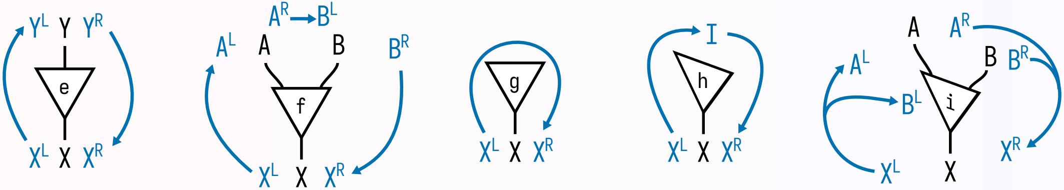
The produoidal category of *spliced monoidal arrows*



Splicing is right adjoint to contouring (II)

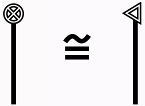
Theorem (E, Hefford, and Román, 2024).

$$\mathsf{MonCat} \underset{\text{splice}}{\overset{\text{contour}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{ProDuo}$$



Normalization

Any produoidal category can be *normalized*: this forces an isomorphism



$$\begin{aligned}
\mathcal{NC}(A;B) &:= \mathcal{C}(N \otimes A \otimes N; B)\\
\mathcal{NC}(A \otimes B;C) &:= \mathcal{C}(N \otimes A \otimes N \otimes B \otimes N; C)\\
\mathcal{NC}(I;A) &:= \mathcal{C}(N; A)\\
\mathcal{NC}(A \triangleleft B;C) &:= \mathcal{C}((N \otimes A \otimes N) \triangleleft (N \otimes B \otimes N); C)\\
\mathcal{NC}(N;A) &:= \mathcal{C}(N; A)
\end{aligned}$$

Theorem (E, Hefford, and Román, 2024). *Normalization extends to an idempotent adjunction between produoidals and normal produoidals.*

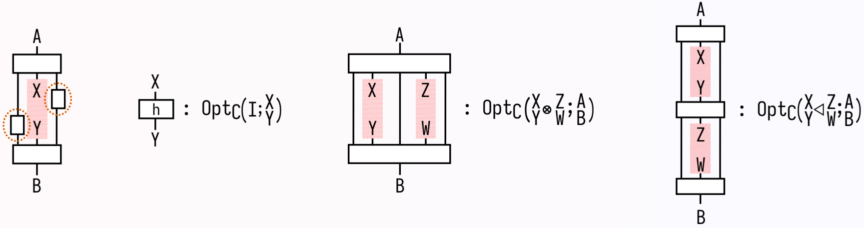
cf. Garner and Franco, 2016 – *Commutativity*

Normalization of spliced monoidal arrows

$$\mathcal{N}\mathsf{Splice}(\mathcal{C})\left(\begin{smallmatrix}X\\Y\end{smallmatrix}; \begin{smallmatrix}A\\B\end{smallmatrix}\right) := \mathsf{Splice}(\mathcal{C})\left(N \otimes \begin{smallmatrix}X\\Y\end{smallmatrix} \otimes N; \begin{smallmatrix}A\\B\end{smallmatrix}\right)$$

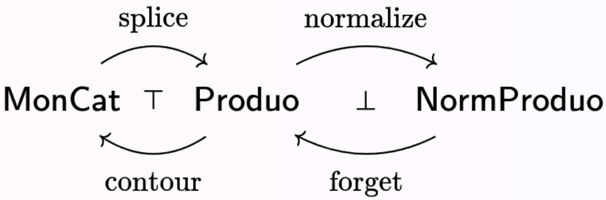
$$= \int^{\begin{smallmatrix}P\\Q\end{smallmatrix}, \begin{smallmatrix}P'\\Q'\end{smallmatrix} \in \mathsf{Splice}(\mathcal{C})} \mathcal{C}(A; P \otimes X \otimes Q) \times \mathcal{C}(P' \otimes Y \otimes Q'; B) \times \mathcal{C}(P; P') \times \mathcal{C}(Q; Q')$$

$$\cong \int^{P, P' \in \mathcal{C}} \mathcal{C}(A; P \otimes X \otimes P') \times \mathcal{C}(P \otimes Y \otimes P'; B)$$



Theorem (E, Hefford, and Román, 2024). *Optics in a monoidal category $\mathbb{C}$ are the free normalization of the cofree produoidal category on $\mathbb{C}$, $\mathsf{Opt}_{\mathcal{C}} \cong \mathcal{N}\mathsf{Splice}(\mathcal{C})$.*

Conclusion and further work



- Duomulticategories
- Tambara modules
- Recovering operads of wiring diagrams
- Polar shuffles

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