Context-Free Languages of String Diagrams

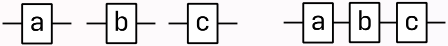
Matt Earnshaw
Tallinn University of Technology

Mario Román
Oxford University

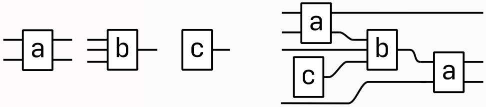
Struture Meets Power Workshop
Tallinn, July 2024

Introduction

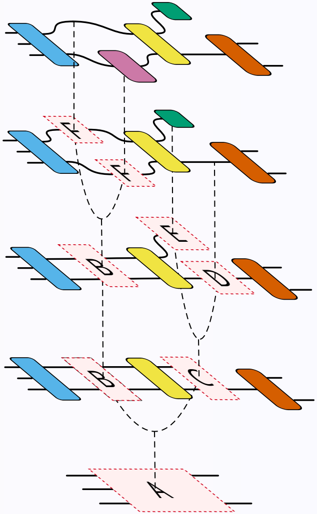
Formal languages of *words* live in monoids:



This talk is about languages that live in an algebraic gadget called *monoidal categories*:

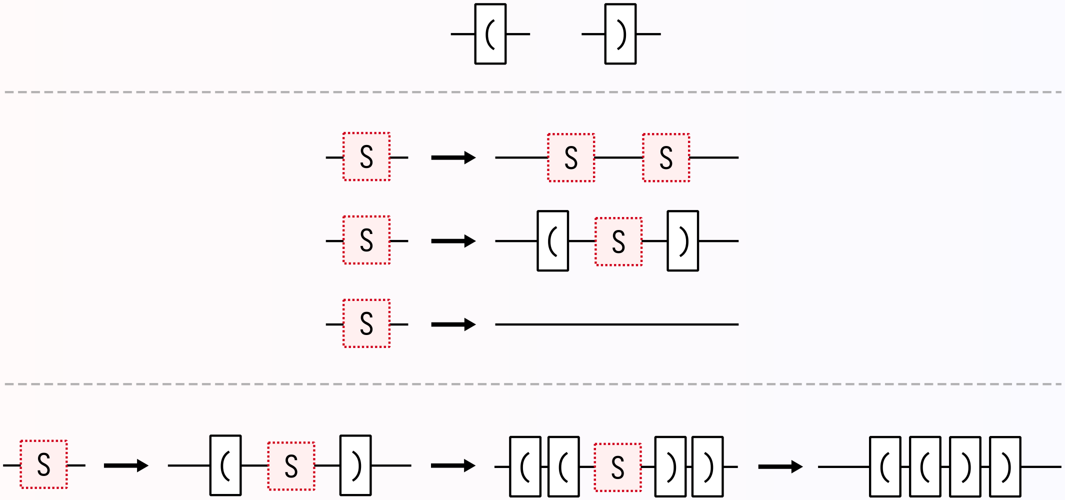


The resulting languages of string diagrams includes languages of words, trees, hypergraphs, and more, and involves some interesting maths.

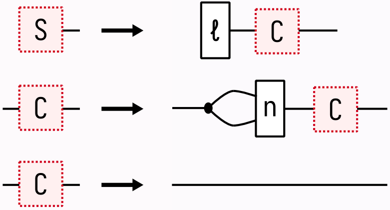


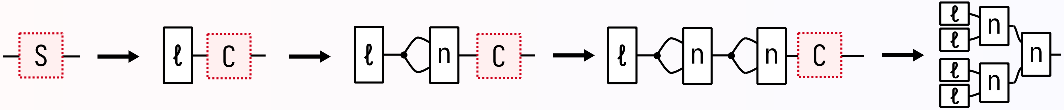
M.E. and M. Román, 2024 – *Context-Free Languages of String Diagrams*

Example: context-free grammars



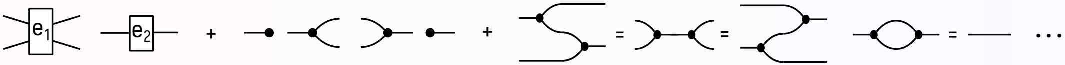
Example: context-free tree grammars

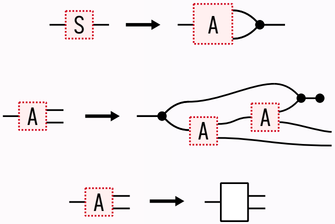


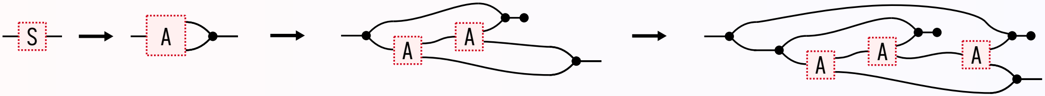


Rounds, 1969

Example: context-free hypergraph (HR) grammars

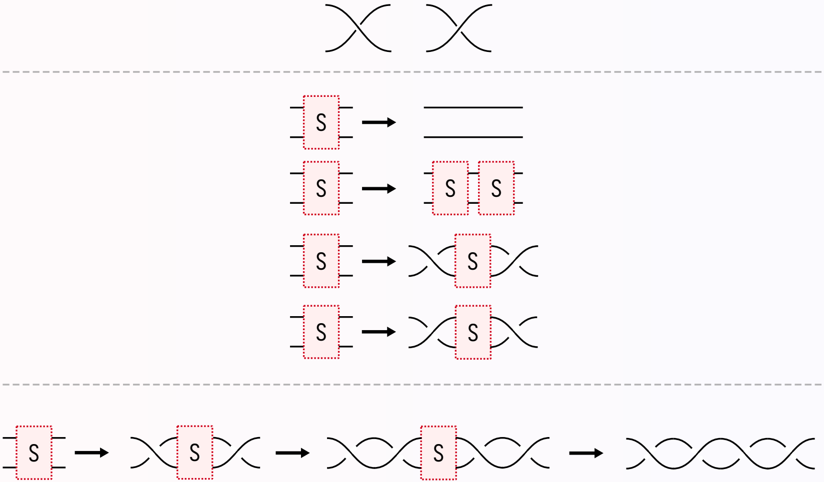




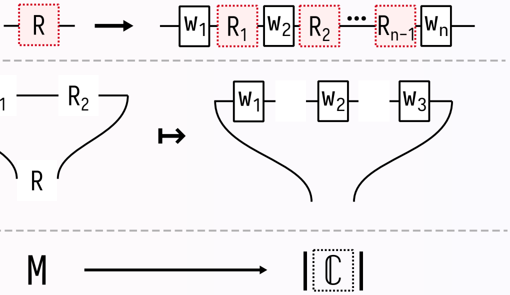


Bauderon and Courcelle, 1987 Habel, 1992

Example: context-free grammar of unbraids



Context-free languages á la Melliès and Zeilberger, 2023; Walters, 1989



For $\mathbb{C}$ a category, $\mathcal{W}\mathbb{C}$ is a multicategory[1] with

- Objects: pairs of objects of $\mathbb{C}$, denoted ${}^{A}_{B}$
- $\mathcal{W}\mathbb{C}(;{}^{X}_{Y}) = \mathbb{C}(X;Y)$,
- $\mathcal{W}\mathbb{C}({}^{A_1}_{B_1},\ldots,{}^{A_n}_{B_n};{}^{X}_{Y}) = \mathbb{C}(X;A_1) \times \prod_{i=1}^{n-1} \mathbb{C}(B_i;A_{i+1}) \times \mathbb{C}(B_n;Y)$,
- composition, splicing into holes using composition in $\mathbb{C}$

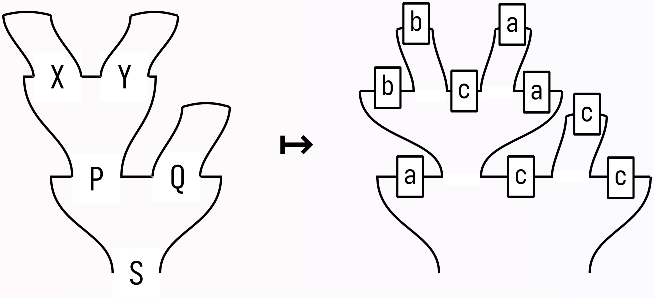
[1] Moreover, a *malleable* multicategory, *cf.* Mario Román, 2023

Context-free languages á la Melliès and Zeilberger, 2023; Walters, 1989

$$M \xrightarrow{\Phi} |\mathbb{C}|$$

$$FM \xrightarrow{\Phi^*} \mathbb{C}$$

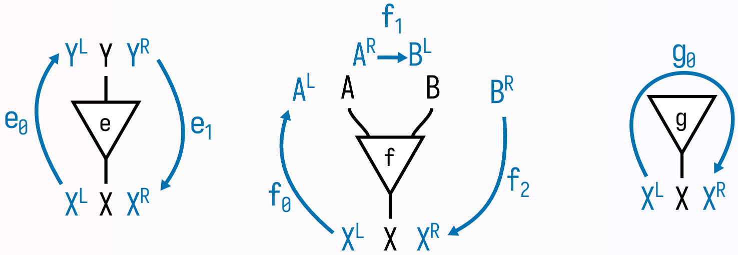
Language of the grammar: $\Phi^*[FM(;S)] \subseteq \mathbb{C}(A;B)$



Context-free languages á la Melliès and Zeilberger, 2023; Walters, 1989

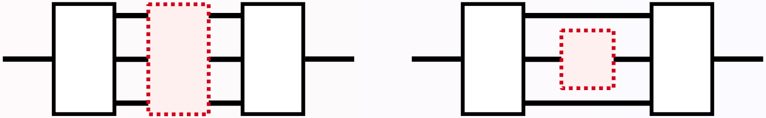
Theorem (Melliès and Zeilberger, 2023)

$\mathcal{W}$: Cat $\to$ MultiCat has a left adjoint given by contours.



Towards context-free monoidal grammars

What is an appropriate multicategory of contexts in a monoidal category?



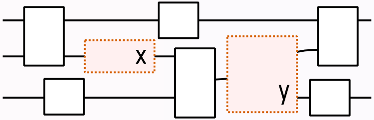
Symmetric multicategory of diagram contexts, $\boxed{\mathbb{C}}$

For any monoidal category $\mathbb{C}$ we can freely add holes of each type

$$\textsc{Identity}\quad \frac{}{\vdash \mathrm{id} : \frac{X}{X}} \qquad \textsc{Generator}\quad \frac{}{\vdash f : \frac{X_1,\dots,X_n}{Y_1,\dots,Y_m}} \qquad \textsc{Hole}\quad \frac{}{\boxed{x} : \frac{A}{B} \vdash \boxed{x} : \frac{A}{B}}$$

$$\textsc{Sequential}\quad \frac{\Gamma \vdash t_1 : \frac{A}{B} \qquad \Delta \vdash t_2 : \frac{B}{C}}{\mathrm{Shuf}(\Gamma;\Delta) \vdash t_1 ; t_2 : \frac{A}{C}} \qquad \textsc{Parallel}\quad \frac{\Gamma \vdash t_1 : \frac{A_1}{B_1} \qquad \Delta \vdash t_2 : \frac{A_2}{B_2}}{\mathrm{Shuf}(\Gamma;\Delta) \vdash t_1 \otimes t_2 : \frac{A_1 {+}{+} A_2}{B_1 {+}{+} B_2}}$$

Derivable term judgements $\Gamma \vdash M : \frac{A}{B}$ up to α-equivalence are *diagram contexts* $\Gamma \to \frac{A}{B}$



$$f : \tfrac{1}{1}, \tfrac{1}{2} \to \tfrac{3}{2}$$

Context-free monoidal grammars

Definition

A *context-free monoidal grammar* over a strict monoidal category $(\mathbb{C}, \otimes, I)$ is a morphism of (symmetric) multigraphs

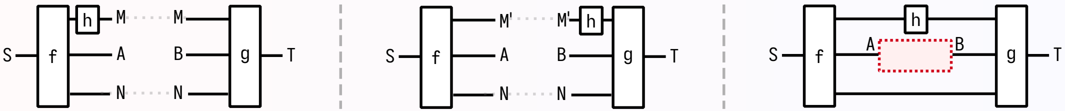
$$\Psi : \mathcal{G} \to | \boxed{\mathbb{C}} |$$

into the underlying multigraph of diagram context in $\mathbb{C}$, where $\mathcal{G}$ is finite, and a start sort $S_{X,Y} \in \Psi^{-1}\binom{X}{Y}$.

By choosing appropriate monoidal categories, we get all the previously shown examples.

Do we also have a left adjoint to forming contexts?

Raw contexts, Raw($\mathbb{C}$)



Proposition

There is an identity on objects multifunctor from raw contexts over $\mathbb{C}$ to diagram contexts over $\mathbb{C}$.

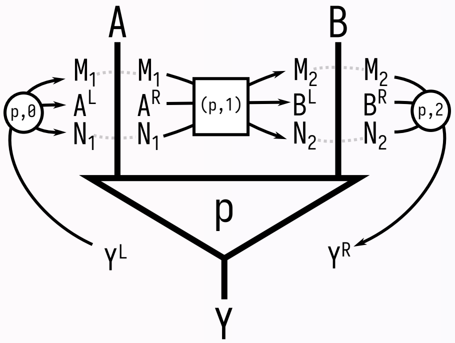
Proposition

Taking raw contexts in a monoidal category extends to a functor $Raw : MonCat \to MultiCat$.

Optical contour

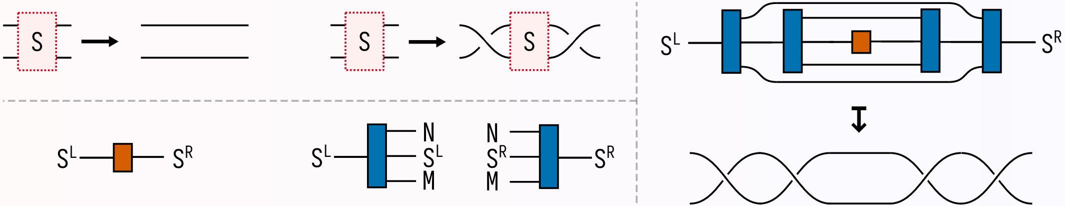
Theorem

Taking raw contexts has a left adjoint, given by the optical contour of a multicategory, $Cont : MultiCat \to MonCat$



A representation theorem for context-free languages of string diagrams

In previous work we introduced the class of *regular* languages of string diagrams (E and Sobociński, 2022). Recognized by automata in which transitions take vectors of states to vectors of states.



Theorem

Every context-free monoidal language arises as the image of a regular monoidal language under a monoidal functor.

References

- Bauderon, Michel and Bruno Courcelle (Dec. 1987). "Graph expressions and graph rewritings". In: *Mathematical systems theory* 20.1, pp. 83–127. ISSN: 1433-0490. DOI: 10.1007/BF01692060. URL: https://doi.org/10.1007/BF01692060.
- E, M. and P. Sobociński (2022). "Regular Monoidal Languages". In: *MFCS 2022*. Vol. 241. LIPIcs, 44:1–44:14. DOI: 10.4230/LIPIcs.MFCS.2022.44.
- Habel, Annegret (1992). *Hyperedge Replacement: Grammars and Languages*. Vol. 643. Lecture Notes in Computer Science. Springer. ISBN: 3-540-56005-X. DOI: 10.1007/BFB0013875. URL: https://doi.org/10.1007/BFb0013875.
- M.E. and M. Román (2024). *Context-Free Languages of String Diagrams*. arXiv: 2404.10653 [cs.FL]. URL: https://arxiv.org/abs/2404.10653.
- Melliès, Paul-André and Noam Zeilberger (Dec. 2023). *The categorical contours of the Chomsky-Schützenberger representation theorem*. URL: https://hal.science/hal-04399404.
- Román, Mario (2023). "Monoidal Context Theory". PhD thesis. Tallinn University of Technology.
- Rounds, William C. (1969). "Context-Free Grammars on Trees". In: *Proceedings of the First Annual ACM Symposium on Theory of Computing*. STOC '69. Marina del Rey, California, USA: Association for Computing Machinery, pp. 143–148. ISBN: 9781450374781. DOI: 10.1145/800169.805428. URL: https://doi.org/10.1145/800169.805428.
- Walters, R.F.C. (1989). "A note on context-free languages". In: *Journal of Pure and Applied Algebra* 62.2, pp. 199–203. ISSN: 0022-4049. DOI: 10.1016/0022-4049(89)90151-5.