ON THE TOPOLOGY OF DIFFERENTIABLE MANIFOLDS

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1. Introduction. This paper is concerned with problems about vector fields in manifolds and sets of vector fields; we may demand that the fields be tangent to the manifold or, if the manifold is in euclidean space, normal to the manifold.

In Part I we study the question of the existence of a field of vectors everywhere normal to a surface in 4-space E^4. It turns out that, although such a field always exists if the surface is orientable, this need not be the case for a non-orientable surface, for example, the projective plane. The proof requires but a minimum knowledge of topology on the part of the reader. In Part II the general theory of sphere bundles (formerly called "sphere spaces") is discussed. In particular, properties of the characteristic classes are studied. For further information see [3].[1] Part III gives applications of the theory to the tangent and normal bundles of a manifold—and hence to problems on vector fields. A formula which may be thought of as a duality theorem plays a fundamental rôle here. The characteristic classes of n-dimensional manifolds M^n with $n = 2, 3, 4$ are studied in particular. The classes of a certain M^3 and of a certain M^4 are determined; it is shown that the latter cannot be imbedded (see § 16 for definitions) in E^7.

PART I. NORMAL VECTOR FIELDS TO SURFACES IN 4-SPACE

2. The 2-dimensional invariant $W^2 \cdot M$. Consider first a surface[2] M in 3-space E^3. At each point $p \in M$ there are two unit

[1] Some further theorems will be found in [5]. The author expects to give a detailed account of the theory in book form.

[2] Only differentiable, or "smooth," manifolds will be considered. Thus M is supposed to have a continuously turning tangent plane.

normal vectors. At a point p_0 choose one of them, $v(p_0)$. We may choose $v(p)$ for all p near p_0 so that $v(p)$ is continuous. But can we so choose $v(p)$ over all of M? Clearly if and only if M is orientable (2-sided in E^3).[3] Thus, for the Möbius strip, if we choose $v(p_0)$ and carry p from p_0 along the strip back to p_0, a continuous $v(p)$, starting as $v(p_0)$, will come back into $-v(p_0)$; hence $v(p)$ cannot be made continuous over the strip.

Now suppose we have $M \subset E^4$. Then at each p there are a tangent plane $P_T(p)$ and a normal plane $P_N(p)$; a normal vector $v(p)$ may lie anywhere in $P_N(p)$. The end points of the unit normals at p form a 1-sphere (i.e. a circle) $S(p)$. Our problem may now be stated as follows: For each $p \in M$ is it possible to choose a point $\phi(p) \in S(p)$ so that ϕ is continuous in M? There is greater likelihood of a positive solution in this case than for $M \subset E^3$, since there is greater freedom of choice for the $v(p)$.

We shall attempt to construct v, or ϕ, over M as follows. Suppose M is triangulated into very small 2-cells, 1-cells, and vertices. Choose $\phi(x_i) \in S(x_i)$ arbitrarily at each vertex x_i. Now take any 1-cell $\sigma^1 = x_ix_j$. This cell is a small slightly curving line segment; the set of $S(p)$ for $p \in \sigma^1$ forms a cylinder $\mathfrak{S}(\sigma^1)$, slightly curved. If we let $\phi(p)$ run along the cylinder from $\phi(x_i)$ to $\phi(x_j)$ as p runs from x_i to x_j, so that $\phi(p) \in S(p)$, then ϕ will be properly defined over σ^1. Thus we define ϕ over all 1-cells.

Now take any 2-cell σ^2. Choose coördinates so that σ^2 is approximately in the (x_1, x_2)-plane; then each $S(p)$ is nearly parallel to the (x_3, x_4)-plane. Hence for each p we may determine the points of $S(p)$ by an angle θ, measuring from the x_3-axis toward the x_4-axis. We now have a $(1-1)$ correspondence between the pairs (p, θ) and the points $\xi(p, \theta)$ of $\mathfrak{S}(\sigma^2)$, where $\mathfrak{S}(\sigma^2)$ is the set[4] of all points on all $S(p)$, $p \in \sigma^2$.

[3] Compare Seifert-Threlfall, *Topologie* (Leipzig, 1934), pp. 272–276. See also [3], pp. 789–791, and formula (20. 4) of the present paper.

[4] Thus this set of points is expressed as the cartesian product of σ^2 and the values θ. Such a "coördinate system" in the $S(p)$ is studied more closely in Part II; see the $\xi_\sigma(p)$.

The values θ we may think of as being points of the unit circle S_0^1 in the (x_3, x_4)-plane. For each $p' \varepsilon S(p)$ let $\xi^{-1}(p, p')$ denote the point θ of S_0^1 such that $\xi(p, \theta) = p'$.

An orientation of E^4 is determined by choosing an ordered set of four independent vectors, w_1, w_2, w_3, w_4, for instance, the unit vectors along a fixed set of axes. We keep this orientation fixed from now on.

Select a definite orientation of σ^2 by choosing an ordered pair of independent vectors u_1, u_2 parallel to σ^2. Then there is a corresponding orientation of S_0^1, by vectors u_3, u_4, such that $u_1, \cdots, u_4$ determine the same orientation of E^4 as $w_1, \cdots, w_4$.

Since ϕ is defined over the boundary $\partial\sigma^2$ of σ^2 we may set

$$\psi(p) = \xi^{-1}(p, \phi(p)), \qquad p \varepsilon \partial\sigma^2. \tag{2. 1}$$

This is a continuous mapping of $\partial\sigma^2$ into S_0^1, giving the "angle" of $\phi(p)$ in $S(p)$. Now $\partial\sigma^2$ is oriented by going from the direction of u_1 toward the direction of u_2. As p runs around $\partial\sigma^2$ once positively, $\psi(p)$ runs around S_0^1 a certain number of times,[5] which number we call $W(\sigma^2)$.

Suppose $W(\sigma^2) = 0$. Then it is not hard to see that it is possible to define $\psi(p)$ through σ^2 so that it is continuous. We may then set

$$\phi(p) = \xi(p, \psi(p)), \qquad p \varepsilon \sigma^2, \tag{2. 2}$$

and thus extend $\phi(p)$ over σ^2. If, however, $W(\sigma^2) \neq 0$, then this is impossible. Of course, altering ϕ on 1-cells of M will change some of the $W(\sigma^2)$. Our problem is, then, *Can the $\phi(p)$ be chosen on the 1-cells so that all $W(\sigma^2) = 0$?* If so, and only if so, $\phi(p)$, and hence the normal vector field $v(p)$, can be defined all over M.

If the surface is open, or has a boundary, then it can be shown that $v(p)$ always exists.[6] Henceforth we consider only closed sur-

[5] This is the "degree" of ψ.

[6] In terms of the general theory the characteristic 2-class, determined by $W^2 \cdot M$ (see below), is a 2-cocycle; but in this case every cocycle is a coboundary. See Part II.

faces. It will turn out that $\sum W(\sigma_i^2)$ is independent of ϕ and is the only invariant; hence v exists if and only if $\sum W(\sigma_i^2) = 0$.

If we let W^2 be the chain whose coefficient on σ_i^2 is $W(\sigma_i^2)$, and let M be the chain $\sum \sigma_i^2$, then[7] (see §§ 9 and 15)

$$\sum W(\sigma_i^2) = W^2 \cdot M. \tag{2.3}$$

3. The invariance of $W^2 \cdot M$. We shall show first that the integer $W(\sigma^2)$ is independent[8] of the orientation chosen for σ^2. For suppose the orientation of σ^2 had been chosen in the opposite manner, say by the ordered pair of vectors u_2, u_1. Then the orientation of S_0^1 would have been chosen in the opposite manner, for instance, by u_4, u_3, to obtain a set u_2, u_1, u_4, u_3 equivalent to $u_1, \cdots, u_4$. In defining $W(\sigma^2)$ we now let p run around $\partial\sigma^2$ in the opposite sense; but as the orientation of S_0^1 is reversed, the same number $W(\sigma^2)$ is determined.

We next consider the effect of altering ϕ to ϕ' on a 1-cell $\sigma^1 = x_i x_j$; let $\phi' = \phi$ elsewhere. There are two 2-cells with σ^1 as face; suppose these are $\sigma_1^2 = x_i x_j x_k$ and $\sigma_2^2 = x_i x_j x_l$; let them be oriented as shown. As p runs along $x_i x_j$, say $\phi'(p)$ runs around the cylinder $\mathfrak{S}(\sigma^1)$ from $\phi'(x_i) = \phi(x_i)$ to $\phi'(x_j) = \phi(x_j)$ a number α of times more than $\phi(p)$. Then, if a fixed circle S_0^1 is chosen as in § 2 and oriented like a circle of $\mathfrak{S}(\sigma^1)$, and ψ_1 and ψ_2 are defined in $\partial\sigma_1^2$ and $\partial\sigma_2^2$ as ψ was defined in § 2, and ψ_1' and ψ_2' are defined with ϕ replaced by ϕ', it is clear that $\psi_h'(p)$ runs around S_0^1 α times more than $\psi_h(p)$ $(h = 1, 2)$. But the chosen orientations of σ_1^2 and σ_2^2 give opposite orientations of the (x_1, x_2)-plane (see § 2); hence, in defining $W(\sigma_1^2)$ and $W(\sigma_2^2)$, S_0^1 is to be given opposite orientations. Therefore one of these $W(\sigma_h^2)$ is increased by α, and the other is decreased by α, so that $\sum W(\sigma_h^2)$ is unchanged.

[7] The chains must be considered chains in the complex associated with M^2 if M^2 is not orientable; see § 15.

[8] In a chain of an ordinary complex $W(-\sigma^2) = -W(\sigma^2)$. The present chain is in a complex K^* whose cells are the pairs (σ, θ), θ being an orientation of the $S(p)$, $p \in \sigma$. We have defined in reality $W(\sigma^2, \theta)$ rather than $W(\sigma^2)$; since $(-\sigma^2, -\theta) = (\sigma^2, \theta)$ (see § 15), $W(-\sigma^2, -\theta) = W(\sigma^2, \theta)$.

Thus altering ϕ on 1-cells has no effect on $W^2 \cdot M$. It is easily seen that the definition of ϕ on the vertices has no effect.[9] It can also be shown that the number $W^2 \cdot M$ comes out the same, no matter how M is subdivided into a complex. Hence it is a topological invariant of the imbedding of M in E^4; it changes sign with reversal of orientation of E^4. Finally, deforming M in E^4 does not alter $W^2 \cdot M$.

If $W^2 \cdot M = 0$, it is fairly simple to show that ϕ may be altered over the 1-cells, one after another, until a ϕ' is obtained, with $W_{\phi'}(\sigma_i^2) = 0$ (all i). Hence *for closed surfaces there exists a normal vector field to M in E^4 if and only if $W^2 \cdot M = 0$.*

4. Types of critical points. Take $M \subset E^4$, and let $p_1, \cdots, p_s$ be the points where the tangent plane is perpendicular to the x_4-direction; we may assume these are finite in number. (If not, a slight deformation of M will give this and the following properties.) Let $f_1(p), \cdots, f_4(p)$ be the coördinates of $p \in M$; set $c_i = f_4(p_i)$. We may assume that the c_i are distinct. Also, for a suitable parametrization (u, v) of the part of M near p_i, f_4 has the form

$$f_4(u, v) = \pm u^2 \pm v^2. \tag{4.1}$$

If both signs are $+$ (or $-$), p_i is a relative minimum (or maximum) of f_4. If one is $+$ and one $-$, we have a saddle point.

Let $M(a, b)$ be the set of all p with $a \leqq f_4(p) \leqq b$. Let $M(a) = M(a, a)$. We may suppose, finally, that for $a \neq$ any c_i, $M(a)$ is a set of smooth closed curves, lying in $E^3(a)$ (the 3-space for which $f_4(p) = a$), and that $M(c_i)$ is a set of closed curves, together with a point p_i (in case of a minimum or a maximum), or a figure 8 through p_i (Fig. 1), or a pair of curves intersecting at right angles at p_i (Fig. 2).

Let c_i' and c_i'' be numbers close to c_i, $c_i' < c_i < c_i''$.

In case of a figure 8, $M(c_i')$ has one (or two) curves near the 8,

[9] If ϕ and ϕ' are defined, we may deform ϕ' into ϕ'', where $\phi'' = \phi$ on the vertices; clearly $W^2_{\phi''} \cdot M = W^2_{\phi'} \cdot M$. By the proof above $W^2_{\phi''} \cdot M = W^2_{\phi} \cdot M$.

while $M(c_i'')$ has two (or one); we then call p of type $1 \to 2$ (or $2 \to 1$). In the case of the two loops, $M(c_i')$ and $M(c_i'')$ have each a single curve near the loops; then p_i is of type $1 \to 1$. A minimum (maximum) is of type $0 \to 1$ $(1 \to 0)$. If p_i is of type $1 \to 1$ (and only then), $M(c_i', c_i'')$ consists of a pair of Möbius strips joining at p_i (Fig. 2).

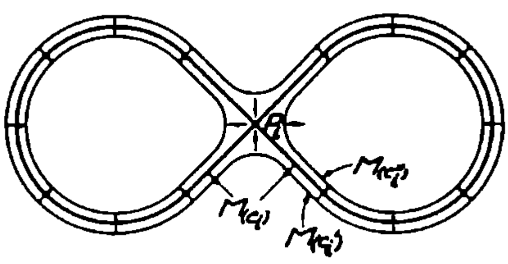

FIG. 1. Point p_i of type $1 \to 2$. Small arrows show direction of increase of f_4

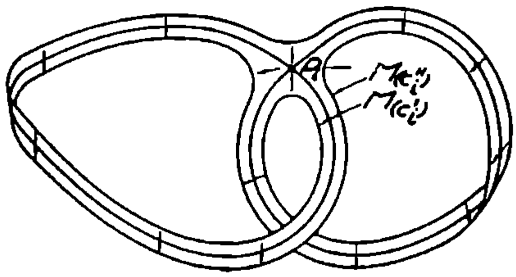

FIG. 2. Point p_i of type $1 \to 1$

5. A method of imbedding M in E^4. If we choose a manner of cutting a surface M into sets $M(a)$ as above, and choose a manner of placing each $M(a)$ in $E^3(a)$ so that the position varies continuously with a, we may thus define an imbedding of M in E^4.

For example, a 2-sphere S^2 may be cut into the north and south poles p_2 and p_1, and circles parallel to the equator. As a increases from c_1 to c_2, $M(a)$ is a curve springing out from p_1, and finally shrinking down rapidly to $p_2 = M(c_2)$.

Note that the part of $M(c_i')$ or $M(c_i'')$ (in E^4) near p_i is approximately an ellipse, or approximately a portion of a hyperbola. (See formula (4. 1).)

If a torus is "stood up on end" in E^4, so that $f_3(p) \equiv 0$, the $M(a)$ are as follows: $M(c_1) = p_1$; $M(c_1'')$ is approximately an ellipse; $M(c_2')$ is a closed curve, two points of which are near together (and near p_2): $M(c_2)$ is a figure 8; $M(c_2'')$ and $M(c_3')$ are pairs of curves; $M(c_3)$ is a figure 8; $M(c_3'')$ and $M(c_4')$ are curves; $M(c_4) = p_4$. The p_i are of types $0 \to 1$, $1 \to 2$, $2 \to 1$, $1 \to 0$.

For the Klein bottle B^2 we may take points p_i of the same types; but we turn over one of the curves of $M(a)(c_2 < a < c_3)$

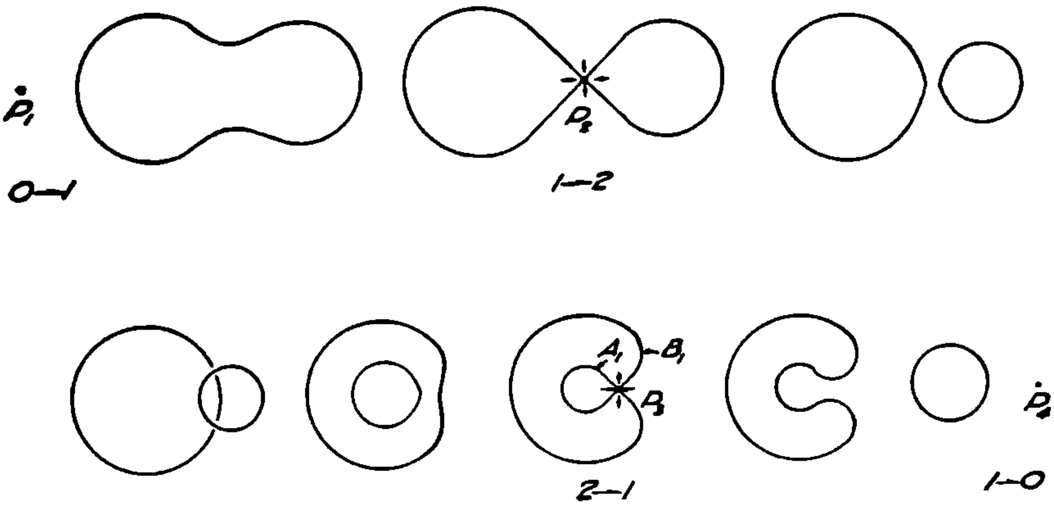

FIG. 3

before bringing it back to the other at p_3, or (see Fig. 3) we carry the curve into an "inside" position before joining it to the other.

Note that each $M(a)$ may be flattened into a plane $E^2(a)$ (the plane of the paper); we thus obtain a smooth mapping of B^2 into 3-space, though of course with singularities.[10] Also, if the $E^2(a)$

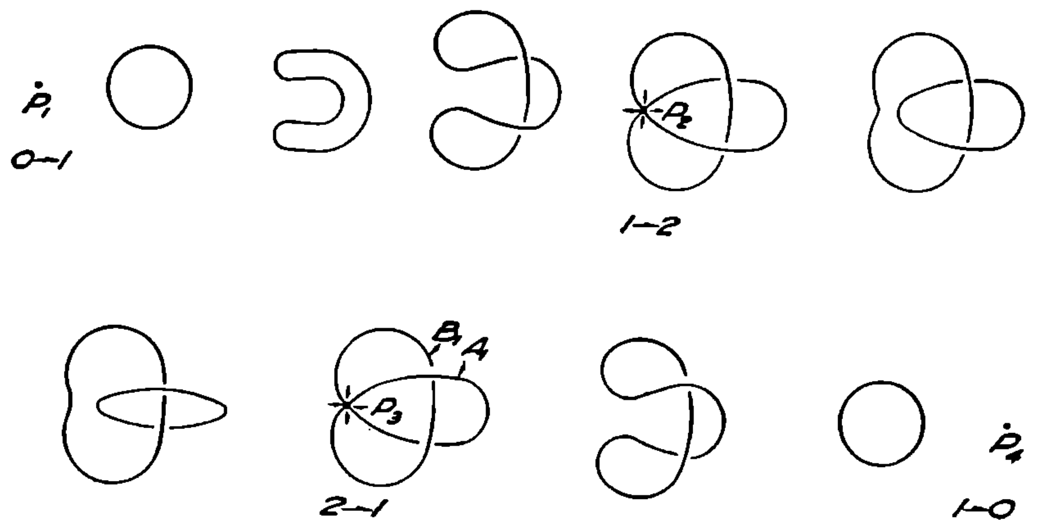

FIG. 4

[10] This may be well seen from the figure in Hilbert and Cohn-Vossen, *Anschauliche Geometrie* (Berlin, 1932), p. 272, considering the x_4-direction as being to the right, or in Seifert-Threlfall, *Topologie*, p. 13, with the x_4-direction down and slightly to the left. We should first smooth off the edge on the bottom of the latter figure.

are parallel to the (x_1, x_2)-plane, then, if $v(p)$ is in the positive x_3-direction at each p, $v(p)$ is never tangent to M, and hence may be deformed into a normal vector field in E^4. Therefore also $W^2 \cdot M = 0$.

In Figure 4 we show an imbedding of B^2 in E^4, for which (see formulas (7. 1) and (6. 5))

$$L_3'' - L_3' = -2[L(A_1', B_1) + L(A_1, B_1')] = -2(1+1) = -4;$$

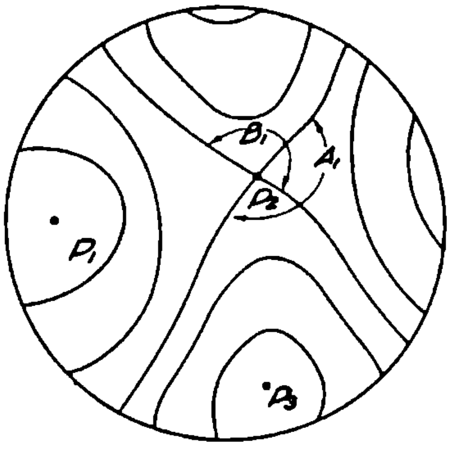

FIG. 5

hence $W^2 \cdot M = -4$. Therefore, in this case, no normal vector field exists.

A projective plane P^2 is formed by identifying opposite points of the circumference of a circular disc. In Figure 5 we show how P^2 may be cut into sets $M(a)$; the p_i are of types $0 \rightarrow 1$, $1 \rightarrow 1$, $1 \rightarrow 0$. Figure 6 shows an "immersion" of P^2 in E^4 with just one singular point, at p^*. We could avoid this singular point by untwisting the curve $M(a)$ for $a < f_3(p^*)$ and shrinking it to a point p_3.

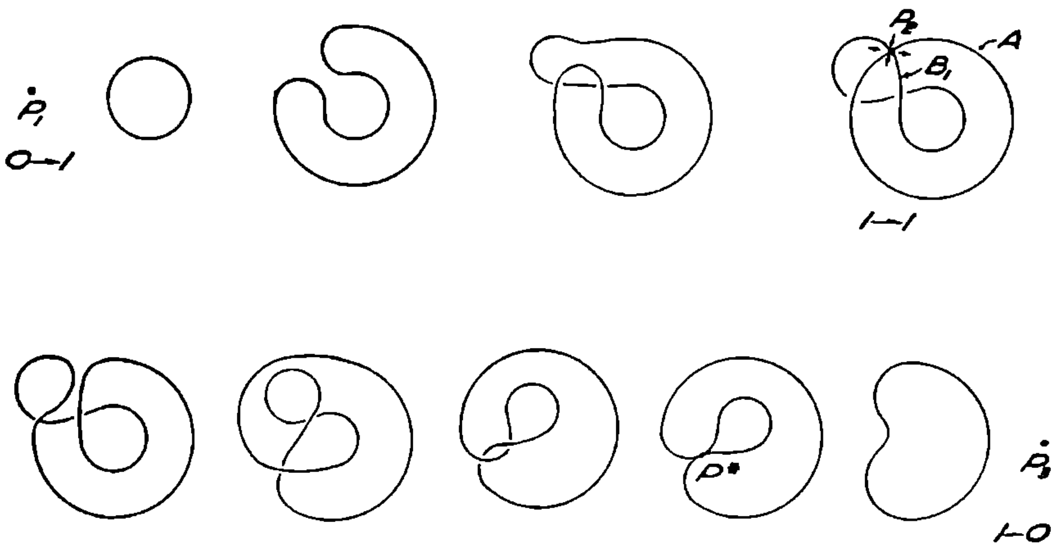

FIG. 6

In the immersion shown we may flatten the $M(a)$ into planes $E^2(a)$, and thus find a normal vector field, as in the case of the Klein bottle.

6. $W^2 \cdot M$ in terms of looping coefficients.[11] Take any $W \subset E^4$. In attempting to define a normal field $v(p)$, or in studying $W^2 \cdot M$, it is sufficient to consider non-tangent fields (see a remark in § 5). It is easy to choose a non-tangent vector field $v(p)$ over each $M(c_i', c_i'')$; moreover, we may let each $v(p)$ lie in $E^3(f_4(p))$. (If $v(p)$ is not tangent to $M(a)$ in $E^3(f_4(p))$ at p, it will not be tangent to M in E^4 at p.) Cut up each $M(c_i', c_i'')$ and each $M(c_i'', c_{i+1}')$ into cells. Considering $v(p)$ as defining $\phi(p)$ on the 1-cells of $M(c_i', c_i'')$, each $W(\sigma_h^2) = 0$ there, as ϕ can be (in fact, is) extended over the interior of σ_h^2. Hence, if we determine

$$(6.1) \qquad D_i = \sum W(\sigma_h^2) \quad \text{over all } \sigma_h^2 \text{ in } M(c_i', c_{i+1}'),$$

then we can find $W^2 \cdot M$ by the formula

$$(6.2) \qquad W^2 \cdot M = \sum_{i=1}^{s-1} D_i.$$

Whenever $v(p)$ is defined over an $M(a)$, and is in $E^3(f_4(p))$, we set $M'(a) =$ all points $p + v(p)$; this is in $E^3(f_4(p))$. Then $M'(a)$ does not intersect $M(a)$, and hence the looping coefficient, $L(M'(a), M(a))$ is defined.

Let $v^*(p) = v(p)$ on $M(c_i'')$. It is easy to extend the definition of $v^*(p)$ continuously through $M(c_i'', c_{i+1}')$ so that it is in $E^3(f_4(p))$, and is not tangent to M. Then clearly

$$(6.3) \qquad L(M^*(c_{i+1}'), M(c_{i+1}')) = L(M'(c_i''), M(c_i'')),$$

where $M^*(c_{i+1}') =$ all $p + v^*(p)$, $p \in M(c_{i+1}')$. Call the last term

[11] If C_1 and C_2 are two oriented non-intersecting closed curves in oriented 3-space, their looping coefficient $L(C_1, C_2)$ is the number of algebraic times that one "loops" about the other. If we let C_1 bound a piece of surface A^2, then L is the number of times C_2 cuts through A^2 in the positive sense, minus the number of times in the negative sense. It can be shown that $L(-C_1, C_2) = -L(C_1, C_2) = -L(C_2, C_1)$. We define $L(\sum C_i, \sum C_j) = \sum L(C_i, C_j)$.

L_i'', and define L_i' similarly. Also, if it should happen that $v^*(p) = v(p)$ in $M(c'_{i+1})$, we would have $D_i = 0$, as is shown by letting $v(p) = v^*(p)$ on the 1-cells of $M(c_i'', c'_{i+1})$. At least we may let $v(p) = v^*(p)$ except interior to 1-cells σ_h^1 of $M(c'_{i+1})$. Then changing $v^*(p)$ to $v(p)$ on σ_h^1 changes L^*_{i+1} by some integer α (which is the number of times $p + v(p)$ runs around σ_h^1 more than $p + v^*(p)$ does), and clearly changes $W(\sigma^2)$ and hence D_i by the same amount, if σ^2 is the 2-cell of $M(c_i'', c'_{i+1})$ with σ_h^1 as a face. Thus we find, if we pay due regard to sign, and use (6. 3),

$$(6.4) \qquad -D_i = L'_{i+1} - L(M^*(c'_{i+1}), M(c'_{i+1})) = L'_{i+1} - L'_i.$$

Summing over i, and noting that there are no curves in $M(c_1')$ or in $M(c_s'')$, we find

$$(6.5) \qquad W^2 \cdot M = \sum_{i=1}^{s-1} (L_i'' - L'_{i+1}) = \sum_{i=1}^{s} (L_i'' - L_i').$$

Observe that changing the orientation of $M(c_i')$ (or $M(c_i'')$) does not alter L_i' (or L_i''); however, we may not reorient only a part of $M(c_i')$ or $M(c_i'')$. Further, corresponding curves of $M(c_i'')$ and $M(c'_{i+1})$ must be similarly oriented for (6. 4) to hold. Of course, $M(a)$ and $M^1(a)$ are to be similarly oriented.

7. Study of the looping coefficients. It is possible to choose certain of the p_i, say $p_{\lambda_1}, p_{\lambda_2}, \cdots$, each of type $1 \to 1$ or $2 \to 1$, and one of the curves, say A_k, of each $M(c_{\lambda_k})$, so that A_k contains p_{λ_k} and so that the A_k form a minimal set of cuts rendering M orientable. That is, there is a curve in M cutting any single A_k which reverses orientation in M, but none cutting no A_k. (All p_i of type $1 \to 1$ must be used.)

For example, for the Klein bottle (see § 5), A_1 is part of the figure 8, $M(c_3)$; for the projective plane, A_1 is one of the loops of $M(c_2)$.

Let B_k be the part of $M(c_{\lambda_k})$ not in A_k. Then A_k and the curves of B_k may be oriented so that the following holds: If the curves of $M(c'_{\lambda_k})$ are oriented like $A_k + B_k$, the curves of $M(c''_{\lambda_k})$

are oriented like $-A_k + B_k$, and the curves of other $M(c_i')$ and $M(c_i'')$ ($i \neq$ any λ_k) are properly, and similarly, oriented, then for each i the curves of $M(c_i'')$ and $M(c_{i+1}')$ are similarly oriented,[12] so that these orientations may be used in the definition of the L_i' and L_i''. Moreover, if $p_i \neq$ any p_{λ_k}, then $M(c_i')$ and $M(c_i'')$ are similarly oriented.

We show now that, if $M(c_i)$ contains no A_k, then $L_i'' = L_i'$. Take a p_i of type $1 \to 2$ or $2 \to 1$. We can pull $M'(c_i')$ into $M'(c_i)$ so that we never intersect $M(c_i')$; hence $L_i' = L(M'(c_i),$

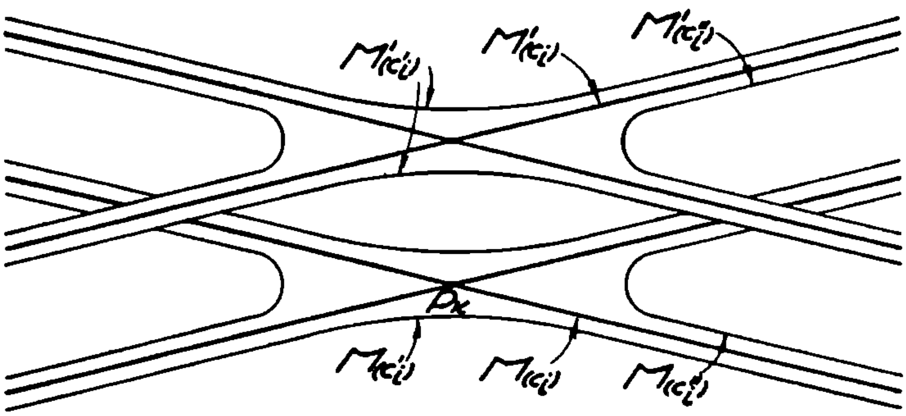

FIG. 7

$M(c_i'))$. Similarly, pull $M'(c_i'')$ into $M'(c_i)$, and $M(c_i')$ and $M(c_i'')$ into $M(c_i)$. (See Fig. 7.) We find thus

$$L_i' = L_i'' = L(M'(c_i), M(c_i)).$$

Clearly $L_i' = L_i''$ if p_i is of type $1 \to 0$ or $0 \to 1$.

Consider next a $p_i = p_{\lambda_k}$. Then the proof above holds, except that we must give the curves corresponding to A_k opposite orientations in $M(c_i')$ and $M(c_i'')$. Hence, by using the bilinearity and commutativity of L,

$$\begin{aligned} L_i'' - L_i' &= L(B_k' - A_k', B_k - A_k) - L(B_k' + A_k', B_k + A_k) \\ &= -2L(B_k', A_k) - 2L(A_k', B_k) \qquad (7.1) \\ &= -2[L(A_k', B_k) + L(A_k, B_k')]. \end{aligned}$$

[12] This may be expressed by the boundary relation $\partial M(c_i'', c_{i+1}') = M(c_{i+1}') - M(c_i'')$. Also $\partial M(c_{\lambda_k}', c_{\lambda_k}'') = M(c_{\lambda_k}) - M(c_{\lambda_k}') + 2A_k$.

Suppose first that p is of type $2 \rightarrow 1$. Then $L(A_k', B_k) = L(A_k, B_k')$, as is seen by pulling B_k away from A_k at p_i and deforming it up to B_k', at the same time pulling A_k' away from B_k' and down to A_k. (For instance, in Figure 7 we might have A_k and A_k' the parts of $M(c_i)$ and $M'(c_i)$ on the right, and B_k and B_k' the parts on the left.) Hence, by (7. 1),

$$(7.\,2)\quad L_i' \;-\; L_i' \equiv 0 \pmod 4 \quad \text{if } p_i \text{ is of type } 2 \rightarrow 1.$$

If p_i is of type $1 \rightarrow 1$, then the parts of A_k and B_k near p_i are smooth curves intersecting at an angle; when we deform B_k into B_k' and A_k' into A_k, the two moving curves cut each other; hence $L(A_k, B_k') = L(A_k', B_k) \pm 1$, and

$$L_i'' - L_i' = -\,2[2L(A_k', B_k) \pm 1],$$

$$(7.\,3)\quad L_i' \;-\; L_i' \equiv 2 \pmod 4 \quad \text{if } p_i \text{ is of type } 1 \rightarrow 1.$$

Let $N_{\alpha\beta}$ be the number of points p_k of type $\alpha \rightarrow \beta$. Then the results above give

$$(7.\,4)\qquad W^2 \cdot M = \sum_{i=1}^{s} (L_i' \;-\; L_i') \equiv 2N_{11} \pmod 4.$$

8. $W^2 \cdot M$ in terms of the characteristic $\chi(M)$ of M. For any complex K, with α^i i-cells ($i = 0, 1, 2, \cdots$), $\chi(M)$ is defined as $\alpha^0 - \alpha^1 + \alpha^2 - \cdots$. This is a topological invariant. If K is cut into disjoint pieces K_i, then $\chi(K) = \sum \chi(K_i)$.

Closed curves and cylinders clearly have the characteristic 0. Hence

$$(8.\,1)\qquad \chi[M(c_i'', c_{i+1}') - M(c_i'')] = 0.$$

It is easily seen that

$$(8.\,2)\quad \chi[M(c_i', c_i'\;) - M(c_i')] = \begin{cases} \;\;1 \text{ if } p_i \text{ of type } 0 \rightarrow 1 \text{ or } 1 \rightarrow 0, \\ -1 \text{ if } p_i \text{ of any other type.} \end{cases}$$

For instance, if p_i is of type $1 \rightarrow 2$, $M(c_i', c_i'')$ is a set of cylinders, together with the part R of $M(c_i', c_i'')$ near p_k; R may be cut into two long strips and a small middle piece; it then has three 2-cells and four inner 1-cells, together with closed curves.

As each point of type $0 \to 1$ or $1 \to 2$ increases the number of curves in $M(a)$ by one while each point of type $2 \to 1$ or $1 \to 0$ decreases the number by one, we have $N_{01} + N_{12} - N_{21} - N_{10} = 0$. Therefore, by adding all relations (8. 1) and (8. 2),

$$\begin{aligned} \chi(M) &= \sum_i \chi[M(c_i', c_i') - M(c_i')] \\ &= N_{01} + N_{10} - (N_{12} + N_{21} + N_{11}) \\ &= 2(N_{01} - N_{21}) - N_{11} \equiv N_{11} \pmod 2). \end{aligned} \tag{8. 3}$$

Multiplying by 2 and comparing with (7. 4) we find

$$W^2 \cdot M \equiv 2\chi(M) \pmod 4. \tag{8. 4}$$

For a non-orientable M with even characteristic there exist imbeddings with $W^2 \cdot M = 0$, and imbeddings with $W^2 \cdot M \neq 0$ (see Figs. 3 and 4). From the proof above we see that, if M is orientable, then $W^2 \cdot M = 0$.[13] Summing up, and using the fundamental theorem on the classification of surfaces, we have:

THEOREM. *To any open surface, or surface with boundary, or any closed orientable surface, in E^4, there exists a normal vector field. Any other surface is homeomorphic to a projective plane P^2, or P^2 with handles, in which case there exists no normal field, or to a Klein bottle B^2, or to B^2 with handles, in which case there may or may not exist a normal field.*[14]

REMARK. For the immersion of P^2 in E^4 with one singularity shown in Figure 6 we noted that a normal vector field existed; hence $W^2 \cdot M = 0$. But for any imbedding without singularity $W^2 \cdot M \neq 0$; hence this singularity cannot be removed by a smooth deformation.

For more details on singularities see § 18.

It seems a reasonable conjecture that for any non-orientable closed surface of characteristic χ the possible values of $W^2 \cdot M$ are

$$2\chi - 4,\ 2\chi,\ 2\chi + 4, \cdots, 4 - 2\chi.$$

[13] Another proof of this will be given in § 18.
[14] This theorem may be proved from general theory also.

See W. S. Massey, Proof of a conjecture of Whitney, Pacific Jour. Math. 31 (1969), 143–156.

PART II. CHARACTERISTIC CLASSES OF SPHERE BUNDLES

9. Homology and cohomology groups. Let K be a fixed complex, with oriented cells σ_i^r. If ${}^r\partial_i^j = [\sigma_j^{r-1} : \sigma_i^r]$ is the incidence number of σ_i^{r-1} and σ_j^r, then the *boundary* and *coboundary* of the chain $A^r = \sum \alpha_i \sigma_i^r$ are defined by

$$(9.1) \qquad \partial A^r = \sum {}^r\partial_i^j \alpha_i \sigma_j^{r-1}, \qquad \delta A^r = \sum {}^{r+1}\partial_j^i \alpha_i \sigma_j^{r+1},$$

where the α_i are elements of an Abelian group G. A^r is a *cycle* or *cocycle* if $\partial A^r = 0$ or $\delta A^r = 0$; A^r is *homologous* to B^r, $A^r \sim B^r$, or *cohomologous* to B^r, $A^r \backsim B^r$, if $A^r - B^r$ is a boundary or a coboundary. The sets of all chains, all cycles, all boundaries, etc., form groups. If we identify homologous r-cycles we obtain the rth *homology group* ${}^G\boldsymbol{H}^r(K)$; similarly for the rth *cohomology group* ${}^G\boldsymbol{H}_r(K)$.

Define *scalar products* of r-chains by

$$(9.2) \qquad \left(\sum \alpha_i \sigma_i^r\right) \cdot \left(\sum \beta_i \sigma_i^r\right) = \sum \alpha_i \beta_i,$$

supposing the multiplication $\alpha\beta$ has meaning. Note that

$$(9.3) \qquad \partial \sigma_i^r \cdot \sigma_j^{r-1} = [\sigma_j^{r-1} : \sigma_i^r] = \sigma_i^r \cdot \delta \sigma_j^{r-1};$$

$$(9.4) \qquad \partial A^r \cdot B^{r-1} = A^r \cdot \delta B^{r-1}; \qquad A^r \cdot B^r = B^r \cdot A^r.$$

The groups G we shall use mostly are the integers, I_0, and the integers mod m, I_m. Let $(a)_m = a_m$ be the integer a reduced mod m; thus 0_2 and 1_2 are the elements of I_2. For an r-I_0-chain $A^r = \sum a^i \sigma_i^r$, define A^r reduced mod m as $(A^r)_m = \sum (a^i)_m \sigma_i^r$.

To each vertex x_i corresponds the chain x_i; we may say x_i has a natural orientation. Set

$$(9.5) \qquad I = \sum x_i; \quad \text{then} \quad \delta I = 0,$$

at least in an ordinary geometric complex. For take any $\sigma^1 = x_i x_j$; then

$$\delta I \cdot \sigma^1 = I \cdot \partial \sigma^1 = I \cdot (x_j - x_i) = 1 - 1 = 0.$$

This need not be true in a general abstract complex; see § 15. Note that, if K is connected, then any 0-I_0-cocycle is $\backsim aI$ for some integer a; hence ${}^{I_0}H_0(K) \approx I_0$ (i.e. is isomorphic to I_0) if K is connected.

Define A^r and B^r to be *orthogonal* if $A^r \cdot B^r = 0$. *Any cocycle*[15] X^r *is orthogonal to any boundary* $A^r = \partial B^{r+1}$; for $X^r \cdot A^r = \partial X^r \cdot B^{r+1} = 0$. Similarly, *any cycle is orthogonal to any coboundary.*

The following theorem may be proved.[16] *An r-I_0-chain X^r is a coboundary if and only if it is orthogonal (mod m) to every r-I_m-cycle A^r for every m* (i.e. $X^r \cdot A^r = 0_m$). *An r-I_m-chain is a coboundary if and only if it is orthogonal to every r-I_m-cycle.* Similarly for boundaries.

There is an important operation converting r-I_m-cocycles X^r into $(r+1)$-I_0-cocycles (or, similarly, r-I_m-cycles into $(r-1)$-I_0-cycles[17]), as follows: Let $\omega_m X^r$ be any r-I_0-cocycle such that $(\omega_m X^r)_m = X^r$. Then $\delta\omega_m X^r \equiv 0 \pmod m$, and hence $Y^{r+1} = (1/m)\delta\omega_m X^r$ exists. The cohomology class of Y^{r+1} is uniquely determined by that of X^r. Note that $mY^{r+1} \backsim 0$.

10. Chains and intersections in manifolds. Let K be a subdivision of the closed manifold M^n. If M^n is orientable,[18] we may orient its n-cells σ_i^n so that $\sum \sigma_i^n$ is a cycle, a *fundamental cycle* of M^n (which we also call M^n); every n-cycle is a multiple of M^n, and hence ${}^{I_0}H^n(M^n) \approx I_0$. Clearly $(M^n)_2$ is a cycle even if M^n is not orientable.

If σ_1^n and σ_2^n have the common face σ^{n-1}, then $\delta\sigma^{n-1} = \pm\sigma_1^n \pm \sigma_2^n$; hence $\sigma_2^n \backsim \pm\sigma_1^n$, and, similarly, each σ_i^n is $\backsim \pm\sigma_1^n$. M^n is non-orientable if and only if $2\sigma_i^n \backsim 0$; clearly ${}^{I_0}H_n(M^n) \approx I_0$ or I_2 ac-

[15] We generally use $X, Y, \cdots$ for cocycles, and $A, B, \cdots$ for cycles.

[16] See, for instance H. Whitney, *On matrices of integers and combinatorial topology*, Duke Math. Journ., 3 (1937), 35–45, Theorem 6. The theorem is due essentially to H. Hopf, *Abbildung der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann., 104, 658.

[17] See Alexandroff-Hopf, *Topologie* (Berlin, 1935), pp. 222–223.

[18] For the non-orientable case see § 15.

cording as M^n is orientable or not. If M^n is oriented, then $\sigma_i^n \smile \sigma_1^n$, and

(10. 1) $\sum \alpha_i \sigma_i^n \smile (\sum \alpha_i)\sigma_1^n, \quad (\sum \alpha_i \sigma_i^n) \cdot M^n = \sum \alpha_i,$

and $X^n \smile Y^n$ if and only if $X^n \cdot M^n = Y^n \cdot M^n$. (This corresponds to the fact that $A^0 \sim B^0$ in a connected complex if and only if $A^0 \cdot I = B^0 \cdot I$.)

Let K be a subdivision of M^n, which we assume closed and oriented, and let K' be the regular subdivision of K. Then the simplexes of K' may be fitted together into cells τ so that to each σ_i^r corresponds a $\tau_i^{n-r} = D(\sigma_i^n)$ cutting through it. Set $D(\tau_i^{n-r}) = \sigma_i^r$. Then

(10. 2) $\partial D(X^r) = (-1)^{r+1} D(\delta X^r), \qquad \delta D(A^r) = (-1)^r D(\partial A^r).$

Hence D, the "dual" operator, maps cycles into cocycles, and conversely, and establishes an isomorphism between ${}^G H^r(M^n)$ and ${}^G \boldsymbol{H}_{n-r}(M^n)$ (Poincaré duality theorem).

Let A^r and B^{n-r} be two singular chains, neither of which meets the boundary of the other. Then their *Kronecker index* $\{A^r, B^{n-r}\}^0$, an integer, is defined and gives their algebraic number of intersections. Thus $\{\sigma_i^r, \tau_i^{n-r}\}^0 = 1$. Let $\{\quad\}_m^0$ mean $\{\quad\}^0$ reduced mod m.

More generally, if $r + s \geqq n$, then $\{A^r, B^s\}$ may be defined as an $(r + s - n)$-chain which is linear in the two variables and satisfies the relation

(10. 3) $\partial\{A^r, B^s\} = (-1)^{n-s}\{\partial A^r, B^s\} + \{A^r, \partial B^s\}.$

Since $\{A^r, B^{n-r}\} \cdot I = \{A^r, B^{n-r}\}^0$, and $\delta I = 0$, this gives

(10. 4) $\{\partial A^r, B^{n-r+1}\}^0 = (-1)^r\{A^r, \partial B^{n-r+1}\}^0.$

In any geometric complex K, bilinear products $X^r \smile Y^s = Z^{r+s}$ may be defined with certain simple properties;[19] in particular,

(10. 5) $\delta(X^r \smile Y^s) = \delta X^r \smile Y^s + (-1)^r X^r \smile \delta Y^s,$

(10. 6) $I \smile X^r = X^r \smile I = X^r.$

[19] See H. Whitney, *On products in a complex*, Ann. of Math., 39 (1938), 397–432. See also [4].

If K is simplicial and its vertices are ordered, $x_1, x_2, \cdots$, we may use

$$(10.7) \qquad (x_{\lambda_0} \cdots x_{\lambda_r}) \smile (x_{\lambda_r} \cdots x_{\lambda_{r+s}}) = x_{\lambda_0} \cdots x_{\lambda_r} \cdots x_{\lambda_{r+s}},$$

for $\lambda_0 < \cdots < \lambda_r < \cdots < \lambda_{r+s}$, and $\sigma^r \smile \sigma^s = 0$ for other pairs of simplexes.

In a manifold, D turns r-cocycles into $(n-r)$-cycles; the product then goes into the Kronecker index in that the same products are defined in the homology groups by the two methods.

Suppose M^{n-r} and M^{n-s} are oriented smooth submanifolds of M^n such that at any point p of intersection, except on an $(n-r-s-1)$-dimensional set, the tangent planes to M^{n-r} and M^{n-s} at p have only a plane P^{n-r-s} in common;[20] then the geometric intersection is a manifold or set of manifolds N^{n-r-s}. Moreover, interpreting these manifolds as chains and orienting N^{n-r-s} properly, we obtain

$$(10.8) \qquad \{M^{n-r}, M^{n-s}\} = N^{n-r-s}.$$

If M^{n-r} and M^{n-s} do not satisfy the given condition, a small deformation of one of them will bring this about; of course the homology class of N is uniquely determined. In particular, $\{M^r, M^r\}$ has meaning for $2r \geqq n$. Note that $\{M^n, M^r\} = \{M^r, M^n\} = M^r$. corresponding to (10.6).

11. Sphere bundles. For any smooth M^2 in E^4 we saw in § 2 how normal 1-spheres $S_N(p)$ were defined; these form the *normal sphere bundle of M^2 in E^4*. At each $p \in M$ there is also a tangent plane, and in it a unit circle $S_T(p)$; these 1-spheres form the *tangent sphere bundle* of M^2. Similarly, for $M^n \subset E^m$ the tangent and normal sphere bundles may be defined; the spheres are of dimension $n-1$ and $m-n-1$ respectively. See also § 19.

We now define a more general concept. A *bundle of spheres* or *sphere bundle* $\mathfrak{B}$ consists of a certain topological space T, the *base*

[20] This will have meaning if we imbed M^n in E^{2n+1}. See §16. The exceptional set may be removed by a slight deformation; see the paper in footnote 33, Theorem 2 (p. 654) and (D) (p. 655).

space (M in the examples above), a set of points $S(p)$ corresponding to each p in M, and certain relations between these sets, as follows: For $p \neq p'$, $S(p)$ and $S(p')$ have no common points. We denote by $\{U_i\}$ a certain set of open sets covering T. For each i and each $p \in U_i$ there is a $(1-1)$ correspondence

$$q' = \xi_{U_i}(p, q) \in S(p), \qquad q \in S_0^\nu, \tag{11. 1}$$

between the unit sphere S_0^ν in $E^{\nu+1}$ and the set of points $S(p)$. Let (11. 1) be written also $q = \xi_{U_i}^{-1}(p, q')$. Set

$$\xi_{U_i,U_j}(p, q) = \xi_{U_i}^{-1}(p, \xi_{U_j}(p, q)), \qquad p \in U_i \cdot U_j. \tag{11. 2}$$

We assume that $\xi_{U_i,U_j}(p)$ is an orthogonal mapping of S_0^ν into itself, which varies continuously with p. The set of all points of all $S(p)$ forms the *total space* $\mathfrak{S}$. It may be made into a topological space in an obvious manner.

Define orthogonality on any $S(p)$ by means of $\xi_{U_i}(p)$ for $p \in U_i$. Since each $\xi_{U_i,U_j}(p)$ is orthogonal, this definition is independent of i.

One may alter the U_i and ξ_{U_i}, obtaining a new sphere bundle. But as long as we obtain a homeomorphic total space, split into spheres in the same fashion, and with the same definition of orthogonality on them we call the bundles *equivalent*, and shall not distinguish between them. Each ξ_{U_i} may be thought of as a coordinate system in that part, $\mathfrak{S}(U_i)$, of $\mathfrak{S}$ for which $p \in U_i$.

We shall deal exclusively with the case that the base space is a complex K. If each closed cell[21] $\bar{\sigma}$ is in some U_i, we can use $\xi_\sigma(p) = \xi_{U_i}(p)$ $(p \in \bar{\sigma})$.

REMARK. If we order the vertices of K, say $x_1, x_2, \cdots$, and for each x_i choose a $U(x_i)$ containing all cells of K with x_i as a vertex, and let $U(\sigma) = U(x_i)$ for any σ whose first vertex is x_i, and use $\xi_\sigma = \xi_{U(\sigma)}$, then, if σ_1 and σ_2 have the same first vertex,

$$\xi_{\sigma_1}(p) = \xi_{\sigma_2}(p), \qquad p \in \sigma_1 \cdot \sigma_2. \tag{11. 3}$$

[21] As a point set, we let σ mean the open cell and $\bar{\sigma}$ the closed cell; $\partial\sigma$ is the point-set boundary of σ.

Similarly we may define a *bundle of fibers* by replacing S_0^ν by any topological space S_0, and the group $G^{\nu+1}$ of orthogonal mappings of S_0^ν into itself by any group of homeomorphisms of S_0 into itself. (See [5].)

The bundle $\mathfrak{B}$ is *simple* if we can choose the ξ_σ so that $\xi_{\sigma,\sigma'}(p)$ = identity always. (This is equivalent to replacing $G^{\nu+1}$ by the group containing the identity alone.) We can then set $\zeta(p) = \xi_\sigma(p)$ ($p \varepsilon \sigma$), and thus express $\mathfrak{S}$ as the Cartesian product $K \times S_0^\nu$. Let $e_1, \cdots, e_{\nu+1}$ be the unit points of the axes of S_0^ν. Then if $\mathfrak{B}$ is simple, $\phi_i(p) = \zeta(p, e_i)$ defines $\nu + 1$ orthogonal *projections* of K into $\mathfrak{S}$. Conversely, if such (continuous) projections exist, we may define $\zeta(p, e_i)$, and hence $\zeta(p, q)$, since orthogonality has meaning in the $S(p)$, and thus prove that $\mathfrak{B}$ is simple. The bundle $\mathfrak{B}$ *is always simple if the base space is a cell.*

12. The characteristic classes, $\nu = 1$. Let $\mathfrak{B}$ be a bundle of 1-spheres (as in Part I). For each vertex x_i of K we may choose a pair of orthogonal points $\phi_1(x_i)$, $\phi_2(x_i)$ in $S(x_i)$. This defines a *characteristic* 1-I_2-*cocycle* $W^1 = W^1_\phi$, as follows. For each 1-cell $\sigma^1 = x_i x_j$ let $W(\sigma^1) = 0_2$ or 1_2 according as ϕ_1 and ϕ_2 define the same or opposite orientations of the ends $S(x_i)$, $S(x_j)$ of the cylinder $\mathfrak{S}(\sigma^1)$; that is, according as the ordered pair of points

$$\psi_{l,\sigma^1}(p) = \overset{-1}{\xi_{\sigma^1}}(p, \phi_l(p)) \qquad (l = 1, 2) \tag{12. 1}$$

of S_0^1 define the same or opposite orientations of S_0^1 for $p = x_i$ as for $p = x_j$. Set

$$W^1 = \sum W(\sigma_h^1)\sigma_h^1 \quad \text{(coefficients in } I_2); \tag{12. 2}$$

this is the required cocycle. For any ϕ', $W^1_{\phi'} \backsim W^1_\phi$; hence the cohomology class $\boldsymbol{W}^1$ of W^1 is invariant; this is the *characteristic* 1-*class* of $\mathfrak{B}$. The bundle is *orientable* if we may make all $\xi_{\sigma,\sigma'}(p)$ rotations; this is so if and only if $\boldsymbol{W}^1 = 0$. (For further details see [3]. The $\boldsymbol{W}^r$ are there called F^r.)

We may not be able to extend the definition of both ϕ_1 and ϕ_2

continuously over the 1-cells of K so that they are orthogonal; certainly not if $W^1 \neq 0$. However, the definition of ϕ_1 can be extended, and this will define a *characteristic 2-I_0-cocycle* $W^2 = W^2_\phi$ as follows.[22] We suppose that $\mathfrak{B}$ is orientable until § 14, and that the $\xi_{\sigma,\sigma'}(p)$ are rotations.

Take any oriented 2-cell $\sigma^2 = x_i x_j x_k$ of K. Set

$$(12.3) \qquad \psi_{\sigma^2}(p) = \overset{-1}{\xi}_{\sigma_2}(p, \phi(p)), \qquad p \, \varepsilon \, \partial\sigma^2;$$

this is a mapping of $\partial\sigma^2$ into S^1_0. Let $W(\sigma^2)$ be the degree of $\psi\sigma_2$ (compare § 2). Then $W^2 = \sum W(\sigma^2_h)\sigma^2_h$; the cohomology class $\mathbf{W}^2$ of W^2 is the characteristic 2-class of $\mathfrak{B}$, and is uniquely determined.

The proof runs as follows: First, it is clear that $W(-\sigma^2) = -W(\sigma^2)$. Now let us find the effect of changing ϕ to ϕ' in $\sigma^1 = x_i x_j$; let $\phi' = \phi$ elsewhere. Then $\phi'(p)$ runs around the cylinder $\mathfrak{S}(\sigma^1)$ some number α of times more than $\phi(p)$ does as p runs from x_i to x_j. That is, if

$$(12.4) \qquad \psi_{\sigma^1}(p) = \overset{-1}{\xi}_{\sigma^1}(p, \phi(p)), \qquad \psi'_{\sigma^1}(p) = \overset{-1}{\xi}_{\sigma^1}(p, \phi'(p)),$$

then $\psi'_{\sigma^1}(p)$ runs around S^1_0 α times more than $\psi_{\sigma^1}(p)$.[23]

Now deform ξ_{σ^1} into $\xi^*_{\sigma^1}$, where $\xi^*_{\sigma^1}(p) = \xi_{\sigma^2}(p)$ $(p \, \varepsilon \, \sigma^1)$, by the lemma below. The integer α is always defined, and is continuous, and hence constant. Therefore $\psi'_{\sigma^2}(p)$ runs around S^1_0 α times more than $\psi_{\sigma^2}(p)$; thus $W_{\phi'}(\sigma^2) - W_\phi(\sigma^2) = \alpha$. If $[\sigma^1 : \sigma^2]$ had been -1 instead of $+1$, we would have found the same relation, with $-\alpha$. Thus in any case, as $[\sigma^1 : \sigma^2] = \delta\sigma^1 \cdot \sigma^2$,

$$W_{\phi'}(\sigma^2) - W_\phi(\sigma^2) = (W^2_{\phi'} - W^2_\phi) \cdot \sigma^2 = \alpha\delta\sigma^1 \cdot \sigma^2,$$

and hence

$$(12.5) \qquad W^2_{\phi'} - W^2_\phi = \delta(\alpha\sigma^1), \qquad W^2_{\phi'} \backsim W^2_\phi.$$

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[22] For an example where W^2 is determined see [3], § 6, (a).

[23] For instance, if $\psi_{\sigma^1}(p)$ runs from e_1 to e_2 and $\psi'_{\sigma^1}(p)$ from e_1 through $-e_2$ and $-e_1$ to e_2, then $\alpha = -1$.

As in § 3, we see that W^2_ϕ is independent of ϕ; it is also independent of the subdivision into a complex K. Because of the following lemma it is independent of the ξ_σ chosen.

LEMMA. *If $u(p) = \xi^{-1}(p)\xi^*(p)$ is a rotation for $p \varepsilon \bar{\sigma}$, then ξ may be deformed into ξ^*.*

The rotation $u(p)$ is an element of the group $\overline{G}^{\nu+1}$ of rotations, and is defined in the closed cell $\bar{\sigma}$; thus it maps $\bar{\sigma}$ into $G^{\nu+1}$. Now $\bar{\sigma}$ may be deformed in itself to a point p_0; if u is applied, $u(p)$ is deformed into $u'(p) = u(p_0)$. Deform $u'(p)$ into $u^*(p) =$ identity. If $u_t(p)$ $(0 \leqq t \leqq 1)$ is the whole deformation, set $\xi_t(p) = \xi^*(p)u_t^{-1}(p)$.

We must still show that W^2_ϕ is a cocycle. Consider any 3-cell σ^3. Set $\phi'(p) = \xi_{\sigma^3}(p, e_1)$ on the 1-cells of σ^3. It may be extended over the rest of the 1-cells of K, and it determines $W^2_{\phi'}$. We showed above that $W^2_{\phi'} - W^2_\phi = \delta X^1$ for some X^1. Clearly $W^2_{\phi'}(\sigma^2_i) = 0$ for all faces σ^2_i of σ^3. Hence $W^2_{\phi'} \cdot \partial\sigma^3 = 0$, and

$$\delta \overset{2}{W}_\phi \cdot \overset{3}{\sigma} = (\delta \overset{2}{W}_{\phi'} - \delta\delta \overset{1}{X}) \cdot \overset{3}{\sigma} = \overset{2}{W}_{\phi'} \cdot \partial \overset{3}{\sigma} = 0.$$

Since σ^3 was arbitrary, $\delta W^2_\phi = 0$.

13. The characteristic classes, general ν. For any integer r, $1 \leqq r \leqq \nu + 1$, a characteristic cohomology class W^r may be defined as follows: Construct a set $\phi = (\phi_1, \cdots, \phi_\alpha)$ $(\alpha = \nu - r + 2)$ of orthogonal projections of K^{r-1} into $\mathfrak{S}$.[24] We construct them over the vertices, 1-cells, etc., in turn. Suppose $\phi_1, \cdots, \phi_{i-1}$ $(i \leqq \alpha)$ have been constructed over K^{r-1}, and ϕ_i over K^{s-1} $(s < r)$. Then ϕ_i is constructed over any σ^s, as follows: For each $p \varepsilon \bar{\sigma}^s$ let $S'(p)$ be the $(\nu - i + 1)$-sphere consisting of all points of $S(p)$ orthogonal to $\phi_1(p), \cdots, \phi_{i-1}(p)$; we must choose $\phi_i(p)$ in $S'(p)$. The $S'(p)$ form a sphere bundle $\mathfrak{B}'$ with $\bar{\sigma}^s$ as base space. Since $\bar{\sigma}^s$ is a cell, $\mathfrak{B}'$ is simple; hence we may choose a coördinate system ζ in it. Set

[24] The symbol K^s denotes the subcomplex of K containing all cells of dimension $\leqq s$.

$$\psi(p) = \zeta^{-1}(p, \phi_i(p)), \qquad p \in \partial\sigma^s;$$

this is a mapping of $\partial\sigma^s$ into S_0'. Now $\partial\sigma^s$ and S_0' are sacks[25] of dimensions $s - 1$ and $\nu - i + 1$ respectively. Since $s - 1 < \nu - i + 1$, it is possible to extend the definition of ψ over σ^s.[26] The projection ϕ_i is extended over σ^s by setting

$$\phi_i(p) = \zeta(p, \psi(p)), \qquad p \in \sigma^s.$$

To define $W^r = W^r_\phi$, take any σ^r and set

$$(13.\ 1) \qquad \psi_{i,\sigma^r}(p) = \overset{-1}{\xi_{\sigma^r}}(p, \phi_i(p)), \quad p \in \partial\sigma^r \qquad (i = 1, \cdots, \alpha).$$

The mapping $\psi_1 = \psi_{1,\sigma^r}$ may be deformed into the mapping $\psi_1^{(1)}(p) = e_1$. It is possible to deform the other $\psi_i(p)$ simultaneously into $\psi_i^{(1)}(p)$, so that the mappings are always orthogonal. Now $\psi_2^{(1)}(p), \cdots, \psi_\alpha^{(1)}(p)$ are in the $(\nu - 1)$-sphere $[e_2, \cdots, e_{\nu+1}]$ of S_0^ν orthogonal to e_1. Deform $\psi_2^{(1)}(p)$ in this sphere into $\psi_2^{(2)}(p) = e_2$, and deform the other $\psi_i^{(1)}(p)$ simultaneously, leaving $\psi_1^{(1)}(p)$ fixed, etc. Finally, $\psi_i^{(i)}(p) = e_i$ $(i < \alpha)$, and $\psi'(p) = \psi_\alpha^{(\alpha-1)}(p)$ is in $[e_\alpha, \cdots, e_{\nu+1}]$, a sphere of dimension $\nu + 1 - \alpha = r - 1$. Since $\partial\sigma^r$ is an $(r - 1)$-sack, ψ' has a degree[27] d. Then $W^r_\phi(\sigma^r) = d$ or d_2, according to the following table:

integers mod 2	if $r = 1$	or $1 < r < \nu + 1$ and r even,
integers	if $r = \nu + 1$	or $1 < r < \nu + 1$ and r odd.

We illustrate why coefficients mod 2 must be used in the case $\nu = 2$, $r = 2$, $\alpha = 2$. The mapping $\psi_1(p)$ may be deformed into $\psi_1^{(1)}(p) = e_1$ in various ways; when $\psi_2(p)$ is deformed simultaneously, the final $\psi'(p)$, mapping $\partial\sigma^2$ into $[e_2, e_3]$, may have different resulting degrees, differing from each other by any multiple of 2. For example, let us identity $\partial\sigma^2$ with $[e_2, e_3]$; suppose

$$\psi_1(p) = p, \qquad \psi_2(p) = -e_1, \qquad p \in \partial\sigma^2.$$

[25] Since we wish to reserve the term *sphere* for a set in which orthogonality is defined, we shall call any set homeomorphic to a sphere a *sack*.

[26] See Alexandroff-Hopf, *Topologie*, pp. 502 and 509.

[27] If $r = 1$, $\sigma^r = x_ix_j$, then $d = 0_2$ or 1_2, according as $\psi'(x_i)$ and $\psi'(x_j)$ are the same or opposite points of $[e_{\nu+1}] = \pm e_{\nu+1}$.

If we deform $\psi_1(p) = p$ directly into e_1, along the shortest arc of the circle $[p, e_1]$ through p and e_1, we may deform $\psi_2(p)$ into $\psi'(p) = p$, along $[p, e_1]$ simultaneously; then $d = 1$. But suppose we first deform $\psi_1(p)$ along $[p, e_1]$ into $\psi_1^*(p) = -e_1$, and deform this along a semicircle of $[e_1, e_2]$ into $\psi_1^{(1)}(p) = e_1$. Then we find $\psi_2^*(p) = -p$, and $\psi'(p) =$ the point of $[e_2, e_3]$ obtained by reflecting p across the x_3-axis; then $d = -1$.

14. Properties of the W^r. We recall that if $\nu = 1$, or if dim $(K) \leqq 3$, then $\mathfrak{B}$ is characterized by the classes W^1 and W^2 (by W^1 alone if $\nu = 0$).[28] This holds even if $\mathfrak{B}$ is not orientable; but see § 15. For $\nu = 2$ and dim $(K) = 4$ this is not true; but $\mathfrak{B}$ can be characterized in case $W^1 = 0$ and $W^2 = 0$, by means of a new invariant.[29]

THEOREM. *If W^{2r} is a characteristic cocycle, then a characteristic cocycle W^{2r+1} may be determined by the formula* (see § 9)

$$W^{2r+1} = \tfrac{1}{2}\delta\omega_2 W^{2r}. \tag{14.1}$$

As a consequence, for any odd r, $2W^r = 0$.

Again $\mathfrak{B}$ need not be orientable. The formula is trivial if $2r = \nu + 1$, for we may then interpret $\omega_2 W^{2r}$ as W^{2r} itself. It holds for $r = 0$ if we interpret W^0 as I in the associated complex, and reduce mod 2 (see § 15).

15. Non-orientable sphere bundles. We first discuss abstract complexes. They have cells and incidence numbers; but the vertices need have no natural orientation. Let us call two complexes *locally isomorphic* if their cells are in $(1 - 1)$ correspondence, and for each σ and corresponding σ^* the complexes σ plus faces and σ^* plus faces are isomorphic. For instance, if K has the cells and boundary relations

$$\partial\sigma_1 = b - a, \qquad \partial\sigma_2 = c - b, \qquad \partial\sigma_3 = a - c,$$

and K^* is similar, except that $\partial\sigma_3^* = -a^* - c^*$, then K and K^*

[28] For this and further theorems see [3], § 7, etc. [29] See [5], § 4.

are locally isomorphic. Thus the corresponding sets σ_3, a, c, and σ_3^*, $-a^*$, c^* of oriented cells form isomorphic complexes. If we demand that any abstract complex be locally isomorphic to a geometric complex, then $\partial\partial A = 0$ and $\delta\delta A = 0$ always.

In the abstract complex K^* let I^* be a "unit 0-chain," i.e. the sum of all arbitrarily oriented vertices. Now δI^* can be $\neq 0$; but it is $\equiv 0 \pmod 2$, so that $W^* = (\frac{1}{2}\delta I^*)_2$ exists. Set

$$W^* = \text{cohomology class of } (\tfrac{1}{2}\delta I^*)_2. \tag{15. 1}$$

This is the *characteristic class* of K^*. Since $\delta(\frac{1}{2}\delta I^*)_2 = (\frac{1}{2}\delta\delta I^*)_2 = 0$, it is a cocycle. Since changing the orientation of a vertex x_i in I^* changes W^* by $(\frac{1}{2}\delta(2x_i))_2 = (\delta x_i)_2$, W^* is an invariant of K^*. *If K_1 and K_2 are locally isomorphic, then they are isomorphic if and only if $W_1 = W_2$.* The complex K is *normal*, i.e. like a geometric complex, if and only if $W = 0$. Given any K and any 1-I_2-cohomology class X in K,[30] there is a locally isomorphic K^* with $W^* = X$.[31] In the example above $(\frac{1}{2}\delta I^*)_2 = (-\sigma_3^*)_2 = (\sigma_3^*)_2$, and $W^* \neq 0$.

Set $\rho(0_2) = 1$, $\rho(1_2) = -1$. If K is simplicial, with ordered vertices, we may define incidences in K^* by letting $[\sigma^{*r-1} : \sigma^{*r}]^* = [\sigma^{r-1} : \sigma^r]$ if σ^{r-1} and σ^r have the same first vertex, while if their first vertices x_i, x_j are distinct, we set

$$[\sigma^{*r-1} : \sigma^{*r}]^* = \rho[W^*(x_i x_j)][\sigma^{r-1} : \sigma^r]. \tag{15. 2}$$

Now take any sphere bundle $\mathfrak{B}$, with base space K. We define the *complex K^* associated with* $\mathfrak{B}$ as follows: For each cell σ of K choose an orientation θ of the spheres $S(p)$, $p \in \bar{\sigma}$; let (σ, θ) be a cell of K^*, where, if $-\theta$ is the orientation opposite to θ, then

$$(\sigma, \theta) = -(-\sigma, \theta) = -(\sigma, -\theta) = (-\sigma, -\theta). \tag{15. 3}$$

Suppose θ and θ' are orientations of a sphere. Then set $(\theta, \theta') = 1$

[30] Clearly any I_2-chain in a complex may be considered a chain in any locally isomorphic complex.

[31] For further details see [4], § 7. The application to manifolds below is due to G. de Rham, *Sur la théorie des intersections et les intégrales multiples*, Comm. Math. Helv., 4 (1933), 151–157.

or -1 according as these orientations agree or not. Now if $\sigma = \sigma^r$ and $\sigma' = \sigma'^{r-1}$ are incident in K, let (σ, θ) and (σ', θ') be incident in K^*, and set

$$(15.4) \qquad [(\sigma', \theta') : (\sigma, \theta)] = (\theta, \theta')[\sigma' : \sigma].$$

The complexes K and K^* are locally isomorphic. For given any σ, we choose a single orientation of all $S(p)$, $p \in \bar{\sigma}$, and use the oriented cells (σ', θ) for faces σ' of σ. Then the incidences in this part of K^* are the same as in K.

If K is the complex defined in the example above, and $\mathfrak{B}$ is the non-orientable 1-sphere bundle over K (whose total space is a Klein bottle), then the associated complex is the K^* defined above.

The complex K^* may also be defined as follows: We may subdivide $\mathfrak{S}$ so that for each σ of K, $\mathfrak{S}(\sigma)$ is a subcomplex of $\mathfrak{S}$, and orient its cells so that $\partial\mathfrak{S}(\sigma) \subset \mathfrak{S}(\partial\sigma)$. Then we may consider each $\mathfrak{S}(\sigma_i^r)$ (which is of dimension $r + \nu$) a cell σ_i^{*r}, as incidence relations are defined for them. These form K^*.

If we choose a fixed set of ξ_σ in $\mathfrak{B}$, then these define orientations of the $S(p)$ $(p \in \bar{\sigma})$. If we identify the cells of K and of K^*, the definition above gives: $[\sigma^{r-1} : \sigma^r]^* = [\sigma^{r-1} : \sigma^r]$ or $-[\sigma^{r-1} : \sigma^r]$ according as $\xi_{\sigma^{r-1}, \sigma^r}(p)$ is a rotation or not.

Consider now the case of the tangent bundle of a manifold M^n. Orienting an $S(p)$ is equivalent to orienting a neighborhood U of the manifold about p. Each n-cell of the associated complex K^* has a natural orientation, namely (σ^n, θ), where θ is the same orientation of U that σ^n gives. The sum $\sum(\sigma_i^n, \theta_i)$ of these cells is a cycle, the *fundamental n-cycle* of K^*. It corresponds to I in a geometric complex. The Poincaré duality theorem holds:

$$(15.5) \qquad {}^G H_p(K) \approx {}^G H^{n-p}(K^*), \quad {}^G H_p(K^*) \approx {}^G H^{n-p}(K).$$

We now show how the characteristic classes of $\mathfrak{B}$ may be determined as cohomology classes in the associated complex K^*. The present definition is preferable even to the one of § 13, since the W^r will not depend on any previously chosen orientation of the $S(p)$. Choose $\phi = (\phi_1, \cdots, \phi_{\nu-r+2})$ over K^{r-1} as before. Given any σ^r, choose an orientation θ of the $S(p)$, $p \in \bar{\sigma}^r$. Write all cells incident with (σ^r, θ) in the form (σ', θ); then the definition of $W(\sigma^r)$ and the derivation of the properties stated go exactly as before. We must note that the definition gives $W(\sigma^r, -\theta) = -W(\sigma^r, \theta)$, as it should.[32]

We now prove

(15. 6) $\qquad W^*(K^*) = W^1(\mathfrak{B})$ if K^* is associated with $\mathfrak{B}$.

Choose orthogonal points $\phi_1(x_i), \cdots, \phi_{\nu+1}(x_i)$ at each vertex x_i. These determine an orientation $\theta_i(x_i)$ of $S(x_i)$. Let $I^* = \sum(x_i, \theta_i)$; then $\delta I^* \cdot (x_i x_j) = 0$ or $\neq 0$ according as θ_i and θ_j agree or disagree in the cylinder $\mathfrak{S}(x_i x_j)$. Thus $(\frac{1}{2}\delta I^*)_2 \cdot (x_i x_j) = W^*(x_i x_j) = 0_2$ or 1_2 depending on which holds. But also $W^1(x_i x_j) = 0_2$ or 1_2 in the same circumstances.

Let K_1, K_2, K_3 be locally isomorphic complexes, with

(15. 7) $\qquad W_1 + W_2 + W_3 = 0.$

Then we may define products $X_1^r \smile Y_2^s = Z_3^{r+s}$, as follows: We suppose the locally isomorphic normal complex K is simplicial, with ordered vertices; then incidences in the K_i may be defined as in (15. 2). We use $W_i = (\frac{1}{2}\delta_i I_i)_2$, and

$$(15.\ 8) \qquad \begin{aligned}(x_{\lambda_0} \cdots x_{\lambda_r})_1 \smile (x_{\lambda_r} \cdots x_{\lambda_{r+s}})_2 \\ = \rho[W_2(x_{\lambda_0} x_{\lambda_r})](x_{\lambda_0} \cdots x_{\lambda_{r+s}})_3,\end{aligned}$$

the subscripts here referring to the complexes used.

Part III. Tangent and Normal Bundles of a Manifold

16. Differentiable manifolds.[33] A *differentiable n-manifold* M^n is a topological space, with a certain system of neighborhoods U_i, each in $(1-1)$ correspondence with the interior Q of the unit

[32] In Part I we used essentially the cells (σ, θ), θ being an orientation of tangent spheres instead of normal spheres. But since $W_T^1 = W_N^1$ (see § 20), the two associated complexes are isomorphic (see (15. 6)).

[33] For a detailed treatment see H. Whitney, *Differentiable manifolds*, Ann. of Math., 37 (1936), 645–680.

sphere in n-space, in such a manner that, if $U_i \cdot U_j \neq 0$, then in the common part the resulting mapping of a portion of Q into another portion is continuously differentiable, with non-vanishing jacobian. By a manifold we shall always mean a differentiable one. It may be shown that any M^n may be *imbedded* in E^{2n+1}, that is, mapped in a $(1-1)$ differentiable manner, so that the matrix of partial derivatives is everywhere of maximum rank. Also, it may be *immersed* in E^{2n}; this is the same as imbedding, except that we allow self-intersections. If M^n is immersed in E^m, we may, by a slight deformation, make the singularities the simplest possible (compare § 18). A differentiable M^n in E^m may be approximated arbitrarily closely by an analytic M^n in E^m. Any M^n may be triangulated into a simplicial complex, with differentiable cells.[34]

17. The characteristic classes of tangent bundles. Let K be a simplicial subdivision of M^n, with simplexes σ_i^r, and let L be the regular subdivision of K. Each vertex y_i^r of L is the center of a simplex σ_i^r of K, and $y_{\lambda_0}^{\alpha_0} y_{\lambda_1}^{\alpha_1} \cdots y_{\lambda_r}^{\alpha_r}$ is a "normally oriented" simplex if $\sigma_{\lambda_{i-1}}^{\alpha_{i-1}}$ is a face of $\sigma_{\lambda_i}^{\alpha_i}$ $(i = 1, \cdots, r)$.

The points p of any simplicial complex K may be given barycentric coördinates $\eta_1(p), \eta_2(p), \cdots$, which are continuous functions of p, and so that if p is interior to the simplex $x_{\lambda_0} \cdots x_{\lambda_r}$, then $\eta_{\lambda_0}, \cdots, \eta_{\lambda_r}$, and only these, are $\neq 0$. Moreover, if we think of each simplex as a flat space, p can be written in the form

$$p = \sum \eta_k(p) x_k, \qquad \eta_k(p) \geqq 0, \qquad \sum \eta_k(p) = 1.$$

In any such complex we may define a continuous vector field by

$$v(p) = \sum_{\lambda < \mu} \eta_\lambda(p) \eta_\mu(p)(x_\mu - x_\lambda). \tag{17.1}$$

This is tangent to K at each p, and vanishes only at the vertices of p. The vectors are shown in Figure 8.

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[34] See S. S. Cairns, *Triangulation of the manifold of class one*, Bull. Amer. Math. Soc., 41 (1935), 549–552.

Define this field in M^n, using L. For example, if $n = 2$, then in a 1-cell of the form $y_i^r y_j^s$ $(r < s)$, $v(p)$ is a positive multiple of $y_j^s - y_i^r$. For $p \varepsilon y_i^0 y_j^1 y_k^2$, $v(p)$ is of the form

$$a(y_j^1 - y_i^0) + b(y_k^2 - y_i^0) + c(y_k^2 - y_j^1),$$

a, b, c being positive. (Compare Stiefel [2].)

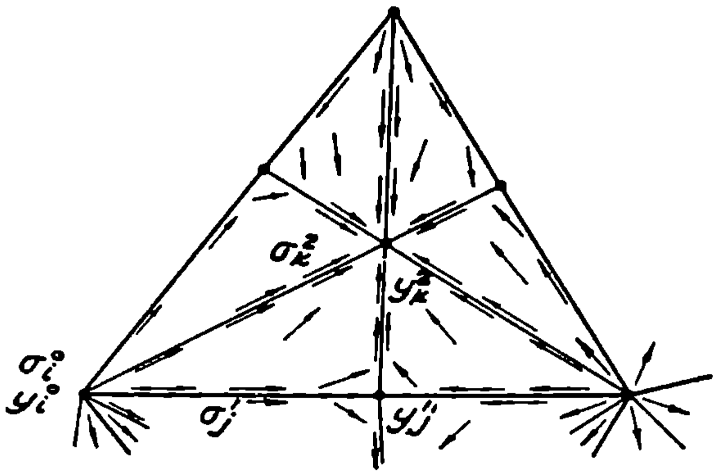

FIG. 8

Let L' be the dual complex of L; its $(n - 1)$-cells cut the 1-cells of L. Since $v(p)$ is defined and $\neq 0$ on L'^{n-1}, we may deform it into a field of unit vectors, and hence define W^n as a cocycle in L' (or in a locally isomorphic complex; see § 15). The dual $D(W^n) = C^0$ of W^n (see § 10) is a cycle in L, a *characteristic* 0-*cycle*.

We shall determine $W^n \cdot M^n = C^0 \cdot I$ in the case $n = 2$. First take a τ_i^2 of L' dual to y_i^0 in L; $W(\tau_i^2) = C(y_i^0)$ is easily seen to be 1 (see Fig. 8); for if we let p run around y_i^0 in a definite direction, $v(p)$ rotates once in the same direction. If p runs around a vertex y_j^1, then $v(p)$ rotates in the opposite direction, so that $C(y_j^1) = -1$. Similarly $C(y_k^2) = 1$. By adding, we obtain

$$C^0 \cdot I = \sum C(y_i^r) = \chi(L),$$

χ being the characteristic of L and hence of M^n (see § 8). The same fact may be proved for general n; hence[35]

[35] This theorem is due to H. Hopf. Compare Alexandroff-Hopf, *op. cit.*, p. 549, and Stiefel, [2], p. 37.

$$W^n \cdot M^n = C^0 \cdot I = \chi(M^n). \tag{17.2}$$

REMARK. If n is odd, then $2W^n$ is 0 (see § 14); hence $2\chi(M^n) = 0$, and $\chi(M^n) = 0$.

REMARK. If we push the points p in the direction of the vectors $v(p)$, we obtain a continuous mapping of M^n into itself, with the fixed points y_i^r; $C^0 \cdot I$ is the sum of the indices of these fixed points.

To determine W^r for $r < n$ we may define a set of $n - r + 1$ vector fields on M^n which are independent except on L^{n-r}; it may be shown that C^{n-r} is the sum of all $(n - r)$-cells of L properly oriented if integer coefficients are used (see [5]).

18. The characteristic classes of normal bundles. Given $M^n \subset M^m$ (compare § 19), we shall derive a formula for W_N^{m-n} of the normal bundle; we may suppose $m \leqq 2n$.

We may take $M^n \subset M^m \subset E^{2m+1}$, and suppose both manifolds are twice differentiable; then the normal spheres $S(p) = S^{m-n-1}(p)$, at present spheres in E^{2m+1}, may be taken so small that after projecting them into M^m they form a "tube" $\mathfrak{S}$ surrounding M^n in M^m.

Let K be a subdivision of M^n, and let ϕ be a projection of K^{m-n-1} into $\mathfrak{S}$. For each σ^{m-n} let $\phi(p)$ be chosen on or inside $S(p)$ in M^m, so that ϕ is continuous; we may make $\phi(p) \neq p$ except at most at a single point of σ^{m-n}. Let K'^{m-n} denote the set of all $\phi(p)$, $p \in K^{m-n}$; this is a deformed position of K^{m-n}. Now the Kronecker index $\{\sigma'^{m-n}, M^n\}^0$ is defined, and it is easily seen that it equals $W_{N,\phi}(\sigma^{m-n})$, reduced mod 2 if necessary; thus $W_{N,\phi}^{m-n}$ is the "intersection chain" of K'^{m-n} with M^n.[36] Clearly, for any chain A^{m-n},

$$W_N^{m-n} \cdot A^{m-n} = \{A'^{m-n}, M^n\}^0. \tag{18.1}$$

Suppose that $m = 2n$. Then this gives

[36] Compare H. Whitney, *On products in a complex*, *loc. cit.*, § 20.

$$(18.2) \qquad W_N^n \cdot M^n = \{M'^n, M^n\}^0.$$

For example, let M^2 be a Möbius strip, and let M^1 be a curve running around it. We may push M^1 into a position M'^1 in M^2 so that it crosses M^1 at just one point. Hence $W_N^1 \cdot M^1 = 1_2$. (As $m - n = 1$, coefficients mod 2 must be used.)

Suppose $M^n \subset E^m$ is orientable. Then M^n may be thought of as a singular cycle bounding a singular chain A^{n+1} in E^m. Now for any $(m-n)$-I_μ-cycle B'^{m-n} of the deformed K'^{m-n} (we use I_2 only if coefficients are mod 2),

$$W_N^{m-n} \cdot B'^{m-n} = \{B'^{m-n}, \partial A^{n+1}\}_\mu^0 = \pm \{\partial B'^{m-n}, A^{n+1}\}_\mu^0 = 0_\mu;$$

hence $W_N^{m-n} \sim 0$, and $W_N^{m-n} = 0$ for the normal bundle of an orientable $M^n \subset E^m$.[37] If M^n is not orientable, then this holds mod 2, but not necessarily otherwise; see Part I.

It is also possible for us to define W_N^r for $r < m - n$ by intersection properties, using the cartesian product of M^n and an $(m-n-r)$-cell.

For $M^n \subset M^m$ we may determine a characteristic cycle $C_N^{n-(m-n)} = C_N^{2n-m}$ directly, as follows: Deform M^n slightly into a position M'^n in M^m as free from intersections with M^n as possible. Then any σ'^{m-n} intersects M^n in at most a single point, as before. The intersections will be along a set of submanifolds M^{2n-m}, which may be taken as formed of cells of the dual subdivision of M^n. If $\tau = \tau_i^{2n-m}$ is dual to $\sigma = \sigma_i^{m-n}$, then τ is in M^{2n-m} if and only if $W_N(\sigma) \neq 0$; thus M^{2n-m}, as a chain, is a characteristic cycle:

$$(18.3) \qquad C_N^{2n-m} = \{M'^n, M^n\}.$$

For example, let M^2 be a Möbius strip in E^3. Then we may deform it away from itself, except along an arc crossing the strip. Then C_N' is this arc (taken mod 2).

Now suppose M^n is immersed in M^m; then we may define the

[37] This is proved slightly differently in [3], p. 795. The proof may be made still more direct with the help of intersection chains; see footnote 36.

(18. 4) *distant intersection* $D\{M^n\}$ of M^n with itself,

as being the set of submanifolds M^{2n-m} along which M^n cuts itself; each part of M^{2n-m} must be considered a chain in each of the two pieces of M^n intersecting there. For instance, for an M^1 in the form of a figure 8 in the plane E^2, $D\{M^1\}$ is a point p, counted once in each piece of the curve; but, as is easily seen, with opposite signs. In fact, *if* $M^n \subset E^{2n}$ *is orientable, and* n *is odd, then* $D\{M^n\} \sim 0$; for each point counts twice, with opposite signs. For M^n orientable or not, and any n, if $M^n \subset E^{2n}$, then $(D\{M^n\})_2 \sim 0$.

For another example, consider the usual immersion of the Klein bottle B^2 in E^3. Here $D\{B^2\}$ (reduced mod 2) is a circle, counted once as a circle on one side of part of the tube, and once as a circle going around another part of the tube. Note that $D\{B^2\}$ is not ~ 0.

If M^n is immersed in M^m, and we wish to determine C_N^{2n-m} as before, we must use only the local intersection, and hence remove the distant intersection. We find, therefore,

$$(18.\ 5)\quad C_N^{2n-m} = \{M'^n,\ M^n\} - D\{M^n\} \quad (M^n \text{ immersed in } M^m).$$

This may be checked for the $B^2 \subset E^3$ above.

19. Product bundles. Take $M^n \subset M^m \subset E^s$. Then at each $p \in M^n$ there are tangent planes $P^m(p)$ and $P^n(p) \subset P^m(p)$, and a normal plane $P^{m-n}(p)$ to $P^n(p)$ in $P^m(p)$. The sphere $S(p) \subset P^m(p)$ we shall call the *join* (or product) of the spheres $S_T(p) = S^{n-1}(p)$ and $S_N(p) = S^{m-n-1}(p)$. Thus the part of the tangent bundle of M^m over M^n is the product of the tangent bundle of M^n and the normal bundle of M^n in M^m. We shall write $\mathfrak{B} = \mathfrak{B}_T \times \mathfrak{B}_N$, or

$$(19.\ 1)\qquad \mathfrak{B}(M^m;\ M^n) = \mathfrak{B}(M^n) \times \mathfrak{B}_N(M^n,\ M^m).$$

In general, given $\mathfrak{B}_1$ and $\mathfrak{B}_2$, each with the base space K, we may define $S(p)$ as the join of $S_1(p)$ and $S_2(p)$, and thus define the product $\mathfrak{B}$, with the base space K.

20. The characteristic classes of a product bundle. Recall that $(a)_2$ is the integer a reduced mod 2; set $(0_2)_2 = 0_2$, $(1_2)_2 = 1_2$. Also set $\omega(0_2) = 0$, $\omega(1_2) = 1$ (we could use any odd number here), and $\omega(a) = a$ for integers a. Set $\omega(\sum a_i\sigma_i^r) = \sum\omega(a_i)\sigma_i^r$, etc. Define

$$X^r \supset Y^s = \omega X^r \smile \omega Y^s. \tag{20. 1}$$

This has topological significance only in certain cases (see [4], § 12); it need not even be a cocycle. Finally, set $W^0 = I = \sum$ vertices. We must be careful about meanings if non-normal complexes are used.

THEOREM. *The characteristic classes of $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2$ are given by the following formulas, according as the coefficient group I_2 or I_0 is used in W^r; in the latter case the chains are chains of the complexes associated with the bundles.*

$$W^r = \begin{cases} \sum (W_1^i \smile W_2^{r-i})_2 & \text{if } I_2 \text{ is used,} \\ \sum W_1^i \supset W_2^{r-i} & \text{if } I_0 \text{ is used.} \end{cases} \tag{20. 2}$$

We shall call this the *duality theorem*; see (21. 9).

If K is a manifold, then we may define characteristic cycles; in the properly chosen complexes (20. 2) gives

$$C^{n-r} = \sum \{C_1^{n-i}, C_2^{n-r+i}\}_2 \quad \text{or} \quad \sum \{\omega C_1^{n-i}, \omega C_2^{n-r+i}\}. \tag{20. 3}$$

We give a few special cases of the theorem:

$$W^1 = W_1^1 \smile W_2^0 + W_1^0 \smile W_2^1 = W_1^1 + W_2^1. \tag{20. 4}$$

For example, if $M \subset E^n$, then $W^1 = 0$, and $W_N^1 = W_T^1$, as is well known.

$$W^2 = (W_1^2)_2 + W_1^1 \smile W_2^1 + (W_2^2)_2 \text{ if } \nu \geqq 2; \tag{20. 5}$$

$$W^2 = W_1^1 \supset W_2^1,\ 2W^2 = 0, \text{ if } \nu_1 = \nu_2 = 0,\ \nu = 1. \tag{20. 6}$$

The W^r for r odd are best found with the help of (14. 1).

21. Formal power series expressions. In this section we shall always reduce mod 2, and use $\equiv$. Write XY for $X \smile Y$, $(X)^2$ for $X \smile X$, etc. Given any $\mathfrak{B}$, define

$$(21.1) \qquad W = \sum W^i t^i, \qquad \overline{W} = \frac{1}{W} = \sum \overline{W}^i t^i;$$

W^0 takes the place of 1 here. Equating coefficients in $W\overline{W} = 1$ gives

$$(21.2) \qquad \overline{W}^0 W^0 \equiv 1, \qquad \overline{W}^0 \equiv W^0 (\equiv 1);$$

$$(21.3) \qquad \overline{W}^1 W^0 + \overline{W}^0 W^1 \equiv \overline{W}^1 + W^1 \equiv 0, \qquad \overline{W}^1 \equiv W^1;$$

$$(21.4) \qquad \overline{W}^2 + \overline{W}^1 W^1 + W^2 \equiv 0, \qquad \overline{W}^2 \equiv W^2 + (W^1)^2;$$

$$(21.5) \qquad \overline{W}^3 + \overline{W}^2 W^1 + \overline{W}^1 W^2 + W^3 \equiv 0, \qquad \overline{W}^3 \equiv W^3 + (W^1)^3;$$

$$(21.6) \qquad \overline{W}^4 \equiv W^4 + (W^2)^2 + W^2 (W^1)^2 + (W^1)^4, \text{ etc.}$$

The duality theorem (mod 2) gives:

$$(21.7) \qquad W \equiv W_1 W_2, \ \overline{W} \equiv \overline{W}_1 \overline{W}_2, \ W_1 \equiv W \overline{W}_2, \ \overline{W}_1 \equiv \overline{W} W_2,$$

etc. Hence, for example,

$$(21.8) \qquad W_1^r \equiv \sum W^i \overline{W}_2^{r-i}, \qquad \overline{W}_1^r \equiv \sum \overline{W}^i W_2^{r-i}.$$

For $M^n \subset E^m$ we have tangent and normal bundles, $\mathfrak{B}_T$ and $\mathfrak{B}_N$. The duality theorem gives, as $W \equiv 1$ for E^m,

$$(21.9) \qquad W_T^r \equiv \overline{W}_N^r, \qquad W_N^r \equiv \overline{W}_T^r \qquad (\text{for } M^n \subset E^m).$$

22. Proof of (20. 6). This case of the duality theorem is relatively simple to prove. We choose it also since it shows why products must be used, and it illustrates the use of non-normal complexes.

Choose coördinate systems $\xi_{\sigma,1}$ for $\mathfrak{B}_1$ so that (11. 3) holds. The system $\xi_{\sigma,1}(p)$ maps e_1 and $-e_1$ (the points of S_0^0) into the two points of $S_1(p)$. Choose $\xi_{\sigma,2}$ for $\mathfrak{B}_2$ so that (11. 3) holds, and so that $\xi_{\sigma,2}(p)$ maps e_2 and $-e_2$ into $S_2(p)$. Then if we map S_0^1 into $S(p)$, $\xi_\sigma(p)$ may be defined as follows:

$$\xi_\sigma(p, ae_1 + be_2) = a\xi_{\sigma,1}(p, e_1) + b\xi_{\sigma,2}(p, e_2) \qquad (a^2 + b^2 = 1);$$

then (11. 3) holds for the ξ_σ.

Set, for each vertex x_i of K,

$$\phi(x_i) = \xi_{x_i,1}(x_i, e_1) = \xi_{x_i}(x_i, e_1), \tag{22. 1}$$

$$\phi'(x_i) = \xi_{x_i,2}(x_i, e_2) = \xi_{x_i}(x_i, e_2) \tag{22. 2}$$

then (compare (12. 1), etc.)

$$\psi_{x_i}(x_i) = e_1, \qquad \psi'_{x_i}(x_i) = e_2. \tag{22. 3}$$

Let these define W_1^1 and W_2^1 respectively. Let S^+ be the upper semicircle of S_0^1, from e_1 through e_2 to $-e_1$. Take any $\sigma^1 = x_ix_j$ $(i < j)$. By (11. 3) and (22. 3) $\psi_{\sigma^1}(x_i) = e_1$. Hence, by definition of $W_1^1(\sigma^1)$, $\psi_{\sigma^1}(x_j) = e_1$, or $-e_1$, according as $W_1^1(\sigma^1) = 0_2$ or 1_2. In the former case set $\phi(p) = \xi_{\sigma^1}(p, e_1)$ $(p \in \sigma^1)$; in the latter case define $\phi(p)$ so that $\psi_{\sigma^1}(p)$ runs across S^+ as p runs from x_i to x_j.

Let ϕ define W^2. Take any $\sigma^2 = x_ix_jx_k$ $(i < j < k)$. We shall show that, if $W_2^1(x_ix_j) = 0_2$ or $W_1^1(x_jx_k) = 0_2$, then $W^2(\sigma^2) = 0$, while if $W_2^1(x_ix_j) = W_1^1(x_jx_k) = 1_2$, then

$$\begin{aligned} W^2(\sigma^2) &= -1 \quad \text{if} \quad W_1^1(x_ix_j) = 0_2, \\ W^2(\sigma^2) &= +1 \quad \text{if} \quad W_1^1(x_ix_j) = 1_2. \end{aligned} \tag{22. 4}$$

This will prove

$$\omega W_2^1(x_ix_j)\omega W_1^1(x_jx_k) = -\rho[W_1^1(x_ix_j)]W^2(\sigma^2),$$

and hence, using (15. 8), we find the required formula,

$$W^2 = -(W_2^1 \cup W_1^1) \smile W_2^1 \cup W_1^1 \smile W_1^1 \cup W_2^1.$$

Now $W(\sigma^2)$ is the degree d of the mapping $\psi_{\sigma^2}(p)$ of $\partial\sigma^2$ into S_0^1. We shall determine d by counting the algebraic number of times that $\psi_{\sigma^2}(p)$ crosses the semicircle S^- opposite S^+; the positive direction of S^- is from $-e_1$ through $-e_2$ toward e_1.

On account of (11. 3) $\psi_{\sigma^2}(p) = \psi_{x_ix_j}(p)$ in x_ix_j, and $= \psi_{x_ix_k}(p)$ in x_ix_k; hence ψ_{σ^2} maps $x_ix_j + x_ix_k$ into S^+. We must still consider x_jx_k.

Suppose $W_2^1(x_ix_j) = 0_2$. Then $\psi'_{x_ix_j}(x_j) = e_2$, i.e. $\xi_{x_ix_j,x_j,2}(x_j, e_2)$

$= e_2$. Since $\xi_{x_ix_j,2} = \xi_{\sigma^2,2}$ and $\xi_{x_j,2} = \xi_{x_jx_k,2}$, this gives $\xi_{\sigma^2,x_jx_k,2}(x_j, e_2) = e_2$, and hence

$$\xi_{\sigma^2,x_jx_k}(p, e_2) = e_2, \qquad p \in x_jx_k.$$

Since $\psi_{x_jx_k}$ maps x_jx_k into S^+, the same is true of

$$\psi_{\sigma^2}(p) = \xi_{\sigma^2,x_jx_k}(p, \psi_{x_jx_k}(p));$$

hence $d = 0$. We suppose now that $W_2^1(x_ix_j) = 1_2$, in which case $\psi_{\sigma^2}(p)$ maps x_jx_k into S^- (or into an end of it).

Suppose that $W_1^1(x_jx_k) = 0_2$. Then as $W_1^1 \cdot \partial\sigma^2 = \delta W_1^1 \cdot \sigma^2 = 0$, $W_1^1(x_ix_k) = W_1^1(x_ix_j)$, and $\psi_{x_ix_k}(x_k) = \psi_{x_ix_j}(x_j)$; hence, by (11. 3), $\psi_{\sigma^2}(x_j) = \psi_{\sigma^2}(x_k)$, so that $\psi_{\sigma^2}(p) =$ constant for $p \in x_jx_k$, and again $d = 0$.

Suppose, finally, that we have $W_2^1(x_ix_j) = W_1^1(x_jx_k) = 1_2$. If $W_1^1(x_ix_j) = 0_2$, then $\psi_{\sigma^2}(x_j) = \psi_{x_ix_j}(x_j) = e_1$, $\psi_{\sigma^2}(x_k) = -e_1$, and $\psi_{\sigma^2}(p)$ runs along S^- from e_1 to $-e_1$; hence $d = -1$. If $W_1^1(x_ix_j) = 1_2$, then $\psi_{\sigma^2}(x_j) = -e_1$, $\psi_{\sigma^2}(x_k) = e_1$, and $\psi_{\sigma^2}(p)$ runs in the other direction, and $d = 1$. This proves (22. 4) and hence (20. 6).

23. The characteristic classes in low-dimensional manifolds. We shall study the characteristic classes of the tangent bundles of manifolds of dimensions 2, 3, and 4. Certain results follow from the fact that M^n may be immersed in E^{2n}. More detailed results follow from a study of the parts of the classes in submanifolds of M^n.

Suppose M^n is imbedded in E^m. Then $(W_N^{m-n})_2 = 0$, by § 18. Hence, by use of (21. 9),

(23. 1) $\quad \overline{W}_T^{m-n} \equiv 0 \quad$ if M^n can be imbedded in E^m.

Recall that M^n can always be immersed in E^{2n}, so that the distant intersection $D\{M^n\}$ is ~ 0 (mod 2) (see § 18); hence $(W_N^n)_2 = 0$, and $\overline{W}_T^n \equiv 0$. Using a remark in § 17, and (21. 4), etc., we have, therefore,[38]

[38] These relations are trivial if M^n is open or bounded; for then the highest cohomology group vanishes.

(23. 2) in any M^2, $W_T^2 + (W_T^1)^2 \equiv 0$;

(23. 3) in any M^3, $(W_T^1)^3 \equiv 0$;

(23. 4) in any M^4, $W_T^4 + (W_T^2)^2 + W_T^2(W_T^1)^2 + (W_T^1)^4 \equiv 0$, etc.

Formulas (23. 2) and (17. 2) show that in any closed M^2, $(W^1 \smile W^1)\cdot M^2 = 0_2$ or 1_2 according as $\chi(M^2)$ is even or odd; hence, if M^2 is orientable, $\chi(M^2)$ is even. Formula (23. 3) gives no information in case M^3 is orientable.

LEMMA. *For any smooth closed curve M^1 in M^2,*

$$\{M'^1, M^1\}_2^0 = W^1 \cdot M^1. \tag{23. 5}$$

For the tangent bundle of M^1 is simple; hence, by (18. 2) and (20. 4),

$$\{M'^1, M^1\}_2^0 = W_N^1 \cdot M^1 = (W^1 + W_T^1)\cdot M^1 = W^1 \cdot M^1.$$

LEMMA. *For any 1-I_2-cocycle X^1 of any M^2,*

$$(X^1 \smile X^1)\cdot M^2 = (X^1 \smile W^1)\cdot M^2. \tag{23. 6}$$

This follows from the last lemma, on using a cycle dual to X, which is homologous to a sum of smooth curves in M^2.

LEMMA. *Any 2 - I_0-cycle (2 - I_2-cycle) in any M^n is homologous to the sum of a set of disjoint closed oriented surfaces (closed surfaces) imbedded in M^n.*

This lemma, due essentially to Stiefel,[39] is proved by considering the cycle a singular cycle, and making small deformations until surfaces are obtained.

THEOREM. *In any M^3,*

$$W^2 = W^1 \smile W^1. \tag{23. 7}$$

[39] Stiefel, [2], p. 48. It would be very useful to obtain such a lemma for r-cycles, $r > 2$. Possibly the use of the lemma can be avoided below by the following method: Define the "characteristic tangent classes" of a cycle by the formulas of § 17; define "characteristic normal classes" of a cycle by formulas of § 18; prove that the duality theorem holds with these definitions.

REMARK. Thus $\overline{W}^2 = 0$, a seemingly stronger condition than (23. 3).

If we use a theorem in § 9 and the last lemma it is sufficient to show that for any closed M^2 in M^3, $W^2 \cdot M^2 = (W^1 \smile W^1) \cdot M^2$. Now $W_N^2 = 0$, as the normal bundle of M^2 in M^3 consists of 0-spheres. Hence, using (20. 5), (20. 4), (23. 2), and (23. 6) with $X^1 = W^1$, we find

$$\begin{aligned} W^2 \cdot M^2 &= [W_T^2 + W_T^1 \smile (W^1 + W_T^1) + W_N^2]_2 \cdot M^2 \\ &= 2(W_T^1 \smile W_T^1) \cdot M^2 + (W^1 \smile W^1) \cdot M^2 = (W^1 \smile W^1) \cdot M^2. \end{aligned}$$

COROLLARY. *In any orientable* M^3, $\mathbf{W}^2 = 0$.

A different proof, not using (20. 3), is given by Stiefel [2, p. 39]. It follows that the tangent bundle of any orientable M^3 is simple (see [2, p. 7] or [3, p. 788]; hence a "parallelism" may be defined in M^3.

THEOREM. *In any orientable* M^4, $\mathbf{W}^3 = 0$.

We shall sketch the proof. By § 9 it is sufficient to prove that for any 3-I_μ-cycle A^3,

$$A^3 \cdot W^3 = A^3 \cdot (\tfrac{1}{2}\delta\omega W^2) = 0_\mu.$$

Choose a 3-I_0-chain B^3 with $(B^3)_\mu = A^3$; say $\partial B^3 = \mu C^2$. Then $A^3 \cdot W^3 = [\frac{1}{2}\mu(C^2 \cdot \omega W^2)]_\mu$; hence it is sufficient to show that $(C^2 \cdot \omega W^2)_2 = 0_2$, i.e. that $C^2 \cdot W^2 = 0_2$.

Let M_1^2, M_2^2, $\cdots$ be disjoint oriented surfaces in M^4 with $C^2 \sim \sum M_i^2$. Since $\mu \sum M_i^2 \sim \partial B^3 \sim 0$, we can choose a singular chain D^3 in M^4 with $\partial D^3 = \sum \mu M_i^2$. Let D_i^3 be the part of D^3 near M_i^2; say $\partial D_i^3 = \mu M_i^2 + E_i^2$. The chain E_i^2 may be considered a cycle in the total space $\mathfrak{S}_N$ of the normal bundle of M_i^2 in M^4.

Let PE_i^2 be the projection of E_i^2 into M_i^2; then $\mu M_i^2 \sim PE_i^2$ in M_i^2. But it may be shown that any $(\nu + 1)$-cycle in a complex K which may be obtained as the projection of a cycle in $\mathfrak{S}$, for a bundle $\mathfrak{B}$ of ν-spheres over K, is orthogonal to $W^{\nu+1}(K)$. Hence $\mu M_i^2 \cdot W_{N,i}^2 = 0$. Since I_0 is used in $W_{N,i}^2$, $M_i^2 \cdot W_{N,i}^2 = 0$. Since M_i^2 is orientable,

$W^1_{T,i} = 0$. Also, by (23. 2), $W^2_{T,i} \cdot M^2_i \equiv 0 \pmod 2$. Therefore, by (20. 5),

$$W^2 \cdot M^2_i = (W^2_{T,i} + W^1_{T,i} \cup W^1_{N,i} + W^2_{N,i})_2 \cdot M^2_i = 0_2.$$

Hence $C^2 \cdot W^2 = 0_2$, as required.

24. Example of an M^3 with $W^2 \not\approx 0$. Let P^2 be the projective plane. Let M^3 be the cartesian product of P^2 and a 1-sphere. We may regard P^2 as imbedded in M^3; then clearly $W^1_N = 0$. Hence $W^1_T = W^1$ over P^2, and since $\chi(P^2) = 1$, (23. 7), (20. 4), (23. 2), and (17. 2) give

$$W^2 \cdot P^2 = (W^1 \cup W^1) \cdot P^2 = (W^1_T \cup W^1_T) \cdot P^2 = W^2_T \cdot P^2 = 1_2;$$

hence $W^2 \not\approx 0$. We could also use (20. 5) to prove this.

We shall study more fully an example to be used in § 25. Consider the cylinder plus interior L^3:

$$x^2 + y^2 \leqq 1, \qquad -1 \leqq z \leqq 1.$$

Now make the following identifications:

$$(x, y, z) = (-x, -y, z) \ (x^2 + y^2 = 1), \ (x, y, -1) = (x, -y, 1).$$

We obtain M^3, which may easily be expressed as a differentiable manifold. The identifications define a continuous mapping f of L^3 into M^3. Define, in L^3, the subsets

$$\text{(24. 1)} \qquad \overline{P}^2 : z = 0; \qquad \overline{Q}^2 : x = 0;$$
$$\overline{S}^1_1 : x = y = 0; \qquad \overline{S}^1_2 : x = z = 0; \qquad \overline{p}_0 : x = y = z = 0.$$

Set $P^2 = f(\overline{P}^2)$, etc. Then P^2 is a projective plane, Q^2 is a Klein bottle, and S^1_1 and S^1_2 are simple closed curves.

We may show that

$$\text{(24. 2)} \qquad \begin{gathered} \{P^2, P^2\}_2 \sim 0, \qquad \{P^2, Q^2\}_2 \sim (S^1_2)_2, \\ \{Q^2, Q^2\}_2 \sim (S^1_1 + S^1_2)_2, \\ \{P^2, S^1_1\}_2 \sim (p_0)_2, \qquad \{P^2, S^1_2\}_2 \sim \{Q^2, S^1_1\}_2 \sim 0, \\ \{Q^2, S^1_2\}_2 \sim (p_0)_2. \end{gathered}$$

For instance, by increasing z we deform $\overline{P}^2$ away from itself; applying f deforms P^2 away from itself; hence $\{P^2, P^2\} \sim 0$. If for $z \geqq \frac{1}{2}$ we twist $\overline{Q}^2$ in one sense through an angle α, and for $z \leqq -\frac{1}{2}$ we twist it in the opposite sense by the same amount, and for $-\frac{1}{2} \leqq z \leqq \frac{1}{2}$ we twist it by an amount varying continuously, then applying f deforms Q^2 into Q'^2; we clearly have $\{Q'^2, Q^2\}_2 = (S_1^1 + S_2^1)_2$.

We shall determine the characteristic cycles C^r of the tangent bundle of M^3. Of course, $C^3 = M^{*3}$, in the associated complex. Also $C^0 = 0$, since 3 is odd. Further,

$$C^2 = (P^2 + Q^2)_2, \qquad C^1 = (S_1^1 + S_2^1)_2. \tag{24. 3}$$

To prove the first relation we note that S_1^1 and S_2^1 both reverse the orientation in M^3; this checks with

$$\{S_1^1, P^2 + Q^2\}_2^0 = \{S_2^1, P^2 + Q^2\}_2^0 = 1_2.$$

The second relation follows from this, (23. 7), and (24. 2). (In the dual form $C^1 = \{C^2, C^2\}$.)

Note that, if the cells of M^3 are properly oriented, we find[40]

$$\partial M^3 = 2(P^2 + Q^2). \tag{24. 4}$$

25. Example of an M^4 with $W^3 = \overline{W}^3 \neq 0$. By (23. 1) this M^4 *cannot be imbedded in E^7.*

First define M_0^3 as in § 24. Determine $\mathfrak{B}_1$ as the 0-sphere bundle with M_0^3 as base space for which

$$C_1^2 = (Q_0^2)_2.$$

Let $\mathfrak{B}$ be the product of $\mathfrak{B}_1$ and a simple 0-sphere bundle $\mathfrak{B}_2$ over M_0^3. We set $M^4 = \mathfrak{S}(\mathfrak{B})$.

Since $\mathfrak{B}_2$ is simple, $f(p) = \xi_{\sigma,2}(p, e_2)$ defines a projection of M_0^3 into M^4; let M^3 be the set $f(M_0^3)$. Use P_0^2 in M_0^3, etc.

Consider the normal bundle $\mathfrak{B}_N(M^3, M^4)$ of M^3 in M^4. For

[40] It may be shown in general that, if $\partial M^n = 2C^{n-1}$, then C^{n-1} is a characteristic cycle, dual to W^1.

any $q \varepsilon M^3$, say $q \varepsilon S(p)$, $p \varepsilon M_0^3$, we may regard the normal 0-sphere $S_N(q)$ of $\mathfrak{B}_N$ as being realized by the pair of points of $S(p)$ orthogonal to q. But these are just the points of $S_1(p)$ of the bundle $\mathfrak{B}_1$; hence we may consider $\mathfrak{B}_N = \mathfrak{B}_1$. In particular,

$$C_N^2(M^3, M^4) = (Q^2)_2. \tag{25. 1}$$

If $C^3(M^4)$ is a characteristic cycle of M^4, dual to W^1, then

$$C^2(M^4; M^3) = \{C^3(M^4), M^3\}$$

is dual, in M^3, to the part of W^1 over M^3. By the duality theorem, (24. 3), and (25. 1),

$$C^2(M^4; M^3) \sim C^2(M^3) + C_N^2(M^3, M^4) = (P^2)_2 \quad \text{in} \quad M^3.$$

From this it follows that

$$C^3(M^4) \sim \mathfrak{S}(P_0^2)_2 \qquad \text{(total space over } P_0^2 \text{ in } \mathfrak{B}). \tag{25. 2}$$

As P_0^2 can be deformed away from itself in M_0^3, so can $\mathfrak{S}(P_0^2)$ in M^4; hence

$$\{C^3(M^4), C^3(M^4)\} \sim 0, \quad W^1 \smile W^1 = 0, \quad (W^1)^3 = 0. \tag{25. 3}$$

Next, since $C_N^1(M^3, M^4) = 0$ (the spheres being 0-spheres), the duality theorem and (25. 1) give

$$\begin{aligned} C^1(M^4; M^3) &\sim C^1(M^3) + \{C^2(M^3), C_N^2(M^3, M^4)\} \\ &\sim (S_1^1 + S_2^1)_2 + \{P^2 + Q^2, Q^2\}_2 \sim (S_2^1)_2. \end{aligned}$$

Hence

$$\{C^2(M^4), M^3\} \sim (S_2^1)_2, \qquad C^2(M^4) \sim \mathfrak{S}(S_{20}^1)_2. \tag{25. 4}$$

We must determine $C^1(M^4)$. Since $W^3(M^4)$ is a cocycle of the complex associated with the tangent bundle of M^4, its dual, $C^1(M^4)$, is a cycle of the geometric complex of M^4 (compare (15. 5)). We determine it by (14. 1) and (25. 4):

$$C^1(M^4) = \tfrac{1}{2}\partial_\omega C^2(M^4) = \tfrac{1}{2}\partial\mathfrak{S}(S_{20}^1).$$

Now in M_0^3,

$$\{S_{20}^1, C^2(\mathfrak{B})\}_2^0 = \{S_{20}^1, C^2(\mathfrak{B}_1) + C^2(\mathfrak{B}_2)\}_2^0 = \{S_{20}^1, Q_0^2\}_2^0 = 1_2;$$

hence the orientation of $\mathfrak{B}$ is reversed over S_{20}^1. Therefore $\mathfrak{S}(S_{20}^1)$ is a Klein bottle, and we may orient its cells so that $\partial\mathfrak{S}(S_{20}^1) = 2\mathfrak{S}(p_{00})$. (This is equivalent to a choice of ω above.) Hence

$$\text{(25. 5)} \qquad C^1(M^4) = \tfrac{1}{2}\partial\mathfrak{S}(S_{20}^1) = \mathfrak{S}(p_{00}).$$

Using (21. 5) and (25. 3) we have

$$\text{(25. 6)} \qquad \bar{C}^1(M^4) = C^1(M^4) = \mathfrak{S}(p_{00}).$$

Since this is not ~ 0 in M^4 (although twice it is), $\bar{C}^1(M^4) \neq 0$, and $\bar{W}^3(M^4) \neq 0$.

REMARK. If M^4 is immersed in E^7, then we may find the distant intersection (mod 2) from (18. 5), (25. 6), and (21. 9); since $\{M'^4, M^4\} \sim 0$ (mod 2),

$$\text{(25. 7)} \qquad \begin{aligned} D\{M^4\} &\equiv \{M'^4, M^4\} + C_N^1(M^4, E^7) \\ &\equiv \bar{C}_T^1(M^4) \equiv \mathfrak{S}(p_{00}). \end{aligned}$$

For further examples see [5].

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LITERATURE

Only the papers most directly connected with the present problem are cited here. Further references may be found in [3].

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