On the sheaf of possible worlds

J. Lambek, McGill University, Montreal[1]

ABSTRACT. We define the spectrum of an intuitionistic type theory $\mathcal{L}$: its points are the saturated prime filters of $\mathcal{L}(V)$, V being a set of variables containing sufficiently many variables of each type A such that $\vdash \exists_{x \in A} \top$, and its basic open sets have the form $V(p) = \{P \in Spec\ \mathcal{L} \mid p \in P\}$, where p ranges over the closed formulas of $\mathcal{L}$. The topos generated by $\mathcal{L}(V)$ is the topos of continuous global sections of a sheaf of "model" toposes, resembling Henkin's non-standard models of $\mathcal{L}$. When $\mathcal{L}$ is "non-constructive", $\mathcal{L}(V)$ may be replaced by the "Hilbert-Bernays completion" of $\mathcal{L}$, essentially obtained from $\mathcal{L}$ by adjoining an indefinite article, whose prime filters are already saturated.

0. Introduction.

The worlds of the title are of course mathematical worlds, inhabited by numbers and functions, not by cabbages and kings. To be precise, they are elementary toposes, in the sense of Lawvere [1972] and Tierney [1972], which resemble the category of sets, intuitionistic analogues of Henkin's [1950] non-standard models.

Before becoming more technical, let me give some historical background. Sheaf representations have been most successful in the theory of commutative rings. Classical results, due to Pierce [1967] and Grothendieck [1960], assert that every commutative regular ring is the ring of continuous sections of a sheaf of fields and, more generally, that every commutative ring is the ring of sections of a sheaf of local rings. The base space of this sheaf is the set of prime ideals, the so-called spectrum $Spec\ R$ of the ring R, endowed with the Stone-Zariski topology: basic open sets have the form

$$V(r) = \{P \in Spec\ R \mid r \notin P\},$$

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where r is any element of R. When R is a regular ring, this can also be written

$$V(s) = \{P \in \ Spec\ R \mid s \in P\},$$

for any element s of R, in view of the identity $r(1-r'r) = 0$. Numerous attempts to extend this theorem to non- commutative rings have not so far resulted in general agreement.

About eight years ago [1981] I realized that there was a corresponding theorem for elementary toposes. The points of the spectrum $Spec\ \mathcal{T}$ of a topos $\mathcal{T}$ were the prime filters of the Heyting algebra consisting of all arrows $p : 1 \to \Omega$ in $\mathcal{T}$, Ω being the usual subobject classifier. As basic open sets I took

$$V(p) = \{P \in \ Spec\ \mathcal{T} \mid p \notin P\}$$

in what I believed to be the correct analogy to rings. However, Ieke Moerdijk, then a student at Amsterdam, persuaded me that a more elegant result could be obtained by taking basic open sets

$$V(p) = \{P \in Spec\ \mathcal{T} \mid p \in P\}.$$

The only drawback of our result [1982] was the fact that the stalks of the sheaf fell short of being the models logicians would have been interested in. It is the purpose of this paper to remedy this.

I wish to take this opportunity to thank Aurelio Carboni for insisting that the stalks of the sheaf should be models. In fact, he has proved such a sheaf representation himself, by completely different methods, without however being sure that his spectrum is a topological space. I also wish to thank Michael Barr and Phil Scott for helpful conversations and the mathematicians of Prague for inviting me to present this work in their beautiful city.

1. Logical and categorical prerequisites.

Let me begin by sketching a modern presentation of type theories; for more details the reader is referred to the book written in collaboration with Phil Scott [1986].

A *type theory* $\mathcal{L}$ consists of types, terms and a deduction relation. We require basic types $1, \Omega$ and N and allow the formation of types $A \times B$ and PA from given types A and B. We admit countably many variables of each type, in addition to which we insist on the following terms:

1	Ω	N	$A \times B$	PA
$*$	$a = a'$	0	$< a, b >$	$\{x \in A \mid \varphi(x)\}$
	$a \in \alpha$	Sn		

Here it is understood that a and a' are given terms of type A, α of type PA, n of type N, b of type B and $\varphi(x)$ of type Ω, the last exhibiting the variable x of type A for purpose of substitution. There is also a deduction symbol $\vdash_X$, X being a finite set of variables, allowing one to write deductions of the form

$$p_1, \ldots p_n \vdash_X p_{n+1},$$

where X contains all the variables occurring freely in the p_i, which are assumed to be terms of type Ω, also called "formulas". Concerning such deductions, one postulates a number of obvious axioms and rules of inference, for example:

$$p \vdash_X p, \qquad \frac{p \vdash_X q}{p \vdash_{X \cup \{y\}} q},$$

$$< a, b > = < c, d > \vdash_X \ a = c, \qquad \frac{\varphi(x) \vdash_{\{x\}} \varphi(Sx)}{\varphi(0) \vdash_{\{x\}} \varphi(x)},$$

where x is a variable of type N. For a complete set of axioms and rules of inference the reader is referred to our book.

Logical symbols are defined as follows:

$$\top \equiv * = *,$$
$$p \wedge q \equiv < p, q > = < \top, \top >,$$
$$p \Rightarrow q \equiv (p \wedge q) = p,$$
$$\forall_{x \in A} \phi(x) \equiv \{x \in A \mid \phi(x)\} = \{x \in A \mid \top\}.$$

It is well-known how other symbols may be defined in terms of the above, thus

$$\perp, -p, \ p \vee q, \ \exists_{x \in A} \phi(x), \ \exists!_{x \in A} \phi(x), \ \{a\}, \ \alpha \subseteq \alpha', \ \alpha \times \beta,$$

and so on. The reader is warned however that our system is intuitionistic, thus $p \vee q$ and $\exists_{x \in A} \phi(x)$ are not defined by De Morgan's Rule but as follows:

$$p \vee q \equiv \forall_{y \in \Omega}(((p \Rightarrow y) \wedge (q \Rightarrow y)) \Rightarrow y),$$
$$\exists_{x \in A} \phi(x) \equiv \forall_{y \in \Omega}(\forall_{x \in A}(\varphi(x) \Rightarrow y) \Rightarrow y).$$

All the usual theorems of intuitionistic higher order logic are obtainable in such a language, which is moreover adequate for intuitionistic arithmetic and analysis. If a classical treatment is required, one merely has to adjoin Aristotle's axiom

$$\vdash \forall_{x \in \Omega}(x \vee -x)$$

and perhaps some type-theoretic equivalent of the axiom of choice.

By a *topos* we shall here understand an elementary topos in the sense of Lawvere and Tierney: a cartesian closed category with a subobject classifier Ω and a natural numbers object N. With any topos $\mathcal{T}$ there is associated a type theory $L(\mathcal{T})$, its *internal language*. The types of this language are the objects of $\mathcal{T}$ and the closed terms of type A are arrows $1 \to A$, 1 being the terminal object. Conversely, with any type theory $\mathcal{L}$ there is associated a topos $T(\mathcal{L})$, the *topos generated* by $\mathcal{L}$. Its objects are closed terms α of type PA in $\mathcal{L}$, where A is any type, and arrows $\alpha \to \beta$, β being of type PB, are given by their *graphs*, namely provably functional relations, that is, closed terms ρ of type $P(A \times B)$ such that in $\mathcal{L}$:

$$\vdash \rho \subseteq \alpha \times \beta \ \wedge \ \forall_{x \in A}(x \in \alpha \Rightarrow \exists!_{y \in B} < x, y > \in \rho).$$

It turns out that every topos is equivalent, as a category, to the topos generated by its internal language: $\mathcal{T} \simeq T(L(\mathcal{T}))$. In particular, it can be assumed without loss in generality that toposes have "canonical subobjects". One could also ensure that $\mathcal{L} \simeq L(T(\mathcal{L}))$, if one adopted a suitably sophisticated definition of morphisms in the "meta-category" of type theories. However, we shall take as morphisms quite naively *translations*, which are assumed to send types to types and terms to terms. As morphisms in the meta-category of toposes one takes *logical functors*, which preserve the logical structure, and which here are also assumed to preserve canonical subobjects. One then finds that T and L become adjoint functors between the meta- category of toposes (with canonical subobjects) and logical functors and the meta-category of type theories and translations. In particular, one has a one-to-one correspondence between translations $\mathcal{L} \to L(\mathcal{T})$ and logical functors $T(\mathcal{L}) \to \mathcal{T}$, either of which may be read as an *interpretation* of $\mathcal{L}$ in $\mathcal{T}$.

While, in some sense, every such topos $\mathcal{T}$ may be regarded as a model of $\mathcal{L}$, one is really interested in those toposes $\mathcal{M}$ which resemble the category of sets inasmuch as $L(\mathcal{M})$ has the following three properties:

(C) not $\vdash \bot$; *(Consistency)*

(DP) if $\vdash p \vee q$ then $\vdash p$ or $\vdash q$; *(Disjunction Property)*

(EP) if $\vdash \exists_{x \in A}\varphi(x)$ then $\vdash \varphi(a)$ for some closed term a of type A. *(Existence Property)*

Under these conditions $\mathcal{M}$ will be called a *model topos*. If $\mathcal{M}$ satisfies merely (C) and (DP), it has been called a *local topos*. The three properties can be translated into algebraic properties of the terminal object 1, as was first pointed out by Peter Freyd:

(C) 1 is not initial,

(DP) 1 is indecomposable,

(EP) 1 is projective.

Properties of the functor $Hom(1,-)$ from $\mathcal{M}$ into the category of sets show that $\mathcal{M}$ is essentially what Henkin calls a non-standard model, at least in the Boolean case, where it is faithful; in fact, this functor preserves finite limits (up to isomorphism), finite coproducts (ditto) and epimorphisms.

We should say a word about the translation $\eta_{\mathcal{L}} \quad \mathcal{L} \to LT(\mathcal{L})$, which constitutes part of the data making T left adjoint to L. To each type A of $\mathcal{L}$ it associates the type $\underline{A} \equiv \{x \in A \mid \top\}$ of $LT(\mathcal{L})$ and to each closed term a of type A in $\mathcal{L}$ it associates the closed term $\underline{a}$ of type $\underline{A}$ in $LT(\mathcal{L})$, namely the arrow $\underline{a} : \underline{1} \to \underline{A}$ in $T(\mathcal{L})$ whose graph is the term $\{< *, a >\}$ of type $P(1 \times A)$. It has been shown [Lambek and Scott 1986, II§14, 3] that $\eta_{\mathcal{L}}$ induces a biunique correspondence between equivalence classes of closed terms of type PA in $\mathcal{L}$ module provable equality and closed terms of type $\underline{PA}$ in $LT(\mathcal{L})$, in particular, that $\eta_{\mathcal{L}}$ is a conservative extension. Moreover [ibidem II §14.8], any type α of $LT(\mathcal{L})$ is an object of $T(\mathcal{L})$, hence a closed term of type PA in $\mathcal{L}$, thus a subtype of $\underline{A} \equiv \{x \in A \mid \top\}$, and we may identify $\{x \in \alpha \mid \varphi(x)\}$ with $\{x \in \underline{A} \mid x \in \underline{\alpha} \wedge \varphi(x)\}$ in $LT(\mathcal{L})$, hence $\exists_{x \in \alpha} \varphi(x)$ with $\exists_{x \in \underline{A}}(x \in \underline{\alpha} \wedge \varphi(x))$.

2. Sheaf representation under Hilbert's Rule.

Given a type theory $\mathcal{L}$ and a filter F of closed formulas (terms of type Ω), one may obtain a type theory $\mathcal{L}/F$; its types and terms are the same as those of $\mathcal{L}$, but there is a new deduction relation, inasmuch as the formulas in F are counted as assumptions. Thus $p \vdash q$ in $\mathcal{L}/F$ means $f,\ p \vdash q$ for some $f \in F$.

Let us now recall the main result of Lambek and Moerdijk [1982]. Given a type theory $\mathcal{L}$, we consider the Heyting algebra of all closed formulas in $\mathcal{L}$. The topological space $Spec\ \mathcal{L}$ has as points the prime filters of the Heyting algebra and as basic open sets the sets

$$V(p) \equiv \{P \in \ Spec\ \mathcal{L} \mid p \in P\},$$

p being any closed formula. These basic open sets are compact. With each $V(p)$ one associates the topos $T(\mathcal{L}/(p))$, where (p) is the principal filter generated by p, thus obtaining a presheaf of toposes. This presheaf turns out to be a sheaf, whose stalks are the local toposes $T(\mathcal{L}/P)$. In particular, $T(\mathcal{L}) \simeq T(\mathcal{L}/(\top))$ is the topos of global sections of a sheaf of local toposes.

It has already been pointed out [Lambek and Scott 1986, II §17] that the stalks are model toposes if $\mathcal{L}$ satisfies the Rule of Choice, equivalently, if all objects of $T(\mathcal{L})$ are projective. Actually, this is true more generally when $\mathcal{L}$ satisfies what we shall call *Hilbert's Rule*:

(H) For any type A such that $\vdash \exists_{x\in A}\top$, if α is a closed term of type PA, then there is a closed term e_α of type A such that $\exists_{x\in A} x \in \alpha \vdash e_\alpha \in \alpha$.

If $\alpha \equiv \{x \in A \mid \varphi(x)\}$, Hilbert would write $e_\alpha \equiv \varepsilon_{x\in A}\varphi(x)$. Note that the hypothesis $\vdash \exists_{x\in A}\top$ is necessary in the type- theoretic version of Hilbert's Rule, because the conclusion implies that $\vdash \exists_{x\in A}(x \in \alpha \Rightarrow e_\alpha \in \alpha)$, from which it follows that $\vdash \exists_{x\in A}\top$. Alternative formulations of Hilbert's Rule will be discussed below.

Incidentally, the assumption $\vdash \exists_{x\in A}\top$ happens to be true for all types in *pure* type theory, where types are defined inductively; but, in the internal language of a topos, it asserts that the arrow $A \to 1$ is an epimorphism, which fails to be so, for example, when A is a proper subobject of 1.

PROPOSITION 2.1 Let $\mathcal{L}$ be a type theory satisfying Hilbert's Rule (H). Then, for any filter F, 1 is projective in $T(\mathcal{L}/F)$. In particular, for any prime filter P, $T(\mathcal{L}/P)$ is a model topos, hence $T(\mathcal{L})$ is the topos of continuous global sections of a sheaf of model toposes.

PROOF. We wish to prove that $LT(\mathcal{L}/F)$ has the existence property. In view of Lemma 2.2 below, it suffices to prove that $\mathcal{L}/F$ has the existence property for all types of the form $A \equiv PB$, for which $\vdash \exists_{x\in A}T$ holds trivially.

Suppose then that α is a closed term of type PA such that $A \equiv PB$ and $\vdash \exists_{x\in A} x \in \alpha$ in $\mathcal{L}/F$, then $f \vdash \exists_{x\in A} x \in \alpha$ in $\mathcal{L}$ for some element f of F. By Hilbert's Rule we obtain $f \vdash e_\alpha \in \alpha$ in $\mathcal{L}$, hence $\vdash e_\alpha \in \alpha$ in $\mathcal{L}/F$. The proof is now complete, except for the following:

LEMMA 2.2. If $\mathcal{L}$ has the existence property for all types $A \equiv PB$, then $LT(\mathcal{L})$ has the existence property for all types whatsoever, that is, the terminal object is projective in $T(\mathcal{L})$.

PROOF. First, consider an existential statement in $LT(\mathcal{L})$ of the special form $\vdash \exists_{x\in \underline{A}} x \in \underline{\alpha}$, where α is a closed term of type $A \equiv PB$ in $\mathcal{L}$, $\underline{A} \equiv \eta_{\mathcal{L}}A$ and $\underline{\alpha} \equiv \eta_{\mathcal{L}}\alpha$. Since $\eta_{\mathcal{L}}$ is a conservative extension, $\vdash \exists_{x\in A} x \in \alpha$ in $\mathcal{L}$.

Since $\mathcal{L}$ is assumed to have the existence property for $A \equiv PB$, there is a closed term a of type A such that $\vdash a \in \alpha$ in $\mathcal{L}$, hence $\vdash \underline{a} \in \underline{\alpha}$ in $LT(\mathcal{L})$.

Now consider an arbitrary existential statement in $LT(\mathcal{L})$, say $\vdash \exists_{y \in \beta} \varphi(y)$, where β is a type in $LT(\mathcal{L})$ which started life as a closed term of type PB in $\mathcal{L}$. As remarked at the end of Section 1, this may be written $\vdash \exists_{y \in \underline{B}}(y \in \underline{\beta} \wedge \varphi(y))$. Now $\{y \in \underline{B} \mid y \in \underline{\beta} \wedge \varphi(y)\}$ is a closed term of type $\underline{PB}$ and so, by the properties of $\eta_{\mathcal{L}}$ discussed at the end of Section 1, it has the form $\underline{\beta'} \equiv \eta_{\mathcal{L}} \beta'$. Our existential statement may now be written $\vdash \exists_{y \in \underline{B}} y \in \underline{\beta'}$ Equivalently,

$$\vdash \exists_{x \in \underline{PB}} \quad (sing\ x \wedge x \subseteq \underline{\beta'}),$$

where $sing\ x \equiv \exists!_{y \in \underline{B}} y \in x$. Take $\underline{\alpha} \equiv \{x \in PB \mid sing\ x \wedge x \subseteq \underline{\beta'}\}$, then we have $\vdash \exists_{x \in \underline{A}} x \in \underline{\alpha}$. In view of the special case of the lemma already established, there is a closed term $\underline{a}$ of type $\underline{A}$ such that $\vdash \underline{a} \in \underline{\alpha}$. Thus $\vdash sing\ \underline{a}$ and $\vdash \underline{a} \subseteq \underline{\beta'}$ in $LT(\mathcal{L})$. As this is the internal language of a topos, there is a closed term $\underline{b}$ of type $\underline{B}$ such that $\vdash \underline{a} = \{\underline{b}\}$ and therefore $\vdash \underline{b} \in \underline{\beta'}$. Note that we are not claiming that $\underline{b}$ has the form $\eta_{\mathcal{L}} b$ for some closed term b of type B in $\mathcal{L}$. Anyway, $\underline{b}$ witnesses the existential statement $\vdash \exists_{y \in \underline{B}} y \in \underline{\beta'}$ and our proof is complete.

The proof depended on the following well-known fact, a proof of which is included for the reader's convenience.

LEMMA 2.3. Suppose $\vdash \exists!_{y \in B} y \in \beta$ in the internal language of a topos $\mathcal{T}$, where $\beta : 1 \to PB$. Then there is a unique arrow $b : 1 \to B$ so that $\vdash b \in \beta$.

PROOF. Let $\iota_B : B \to PB$ be the singleton morphism. Thus we are given $\vdash \exists_{y \in B}(\iota_B y = \beta)$. Since ι_B is a monomorphism, let $h : PB \to \Omega$ be its characteristic morphism, so ι_B is the pullback of $\top \quad 1 \to \Omega$ along h. In particular, $h\iota_B. = .\top\circ_B$, where $. =$ denotes "external" equality, and so $\vdash \exists_{y \in B}(\top = h\beta)$. Therefore $\vdash \top = h\beta$ and so $h\beta. = \top$ By the pullback property, there is a unique arrow $b : 1 \to B$ such that $\iota_B b. = .\beta$, that is, $\vdash \{b\} = \beta$, that is, $\vdash b \in \beta$, as was to be proved.

Let us take a closer look at Hilbert's Rule. It is clearly equivalent to the conjunction of the following two properties, assumed to hold for all types A and all closed terms α of type A:

(EP) if $\vdash \exists_{x \in A} x \in \alpha$ then $\vdash a \in \alpha$ for some closed term a of type A,

(NC) if $\vdash \exists_{x \in A} \top$ then $\vdash \exists_{x \in A}(\exists_{x \in A} x \in \alpha \Rightarrow x \in \alpha)$.

(EP) is what we had called the "existence property". The *non-constructive* property (NC) is a consequence of the Aristotelian axiom $\vdash \forall_{x\in\Omega}(x \vee -x)$ and thus holds in any classical type theory. It is not acceptable to constructivists, as my be seen by taking $A \equiv P(PN)$ and α the set of non-principal ultrafilters in PN.

(NC) may be strengthened to the following assertion called *independence of premises*, which is still a consequence of Hilbert's Rule (H).

(IP) for any closed formula p and any closed term α of type PA, if $\vdash p \Rightarrow \exists_{x\in A} x \in \alpha$ and $\vdash \exists_{x\in A}\top$, then $\vdash \exists_{x\in A}(p \Rightarrow x \in \alpha)$.

Clearly, (NC) may be obtained from (IP) by taking $p \equiv \exists_{x\in A} x \in \alpha$. On the other hand, to derive (IP) from Hilbert's Rule, assume $\vdash \exists_{x\in A}\top$ and $\vdash p \Rightarrow \exists_{x\in A} x \in \alpha$; then $\vdash p \Rightarrow e_\alpha \in \alpha$ by (H), hence $\vdash \exists_{x\in A}(p \Rightarrow x \in \alpha)$ by existential generalization. Thus (H) is also equivalent to the conjunction of (EP) and (IP).

If $\mathcal{L} = L(\mathcal{T})$ is the internal language of a topos $\mathcal{T}$, Hilbert's Rule for $\mathcal{L}$ is equivalent to saying that in $\mathcal{T}$ all subobjects of 1 are projective. Johnstone [1977, §5.2] lists as examples of such toposes all functor categories Sets $^{\mathcal{P}}$, where $\mathcal{P}$ is any well-ordered set, and points out that these do not necessarily satisfy the Rule of Choice. They are usually not even Boolean.

3. Sheaf representation for non-constructive type theories.

In this section we shall ask: if $\mathcal{L}$ satisfies (NC), how close is it to satisfying Hilbert's Rule? The following proposition will imply that some conservative extension of $\mathcal{L}$ satisfies Hilbert's Rule (H).

PROPOSITION 3.1. Every type theory $\mathcal{L}$ has a conservative extension $\mathcal{L}^B$, its Hilbert-Bernays completion, with the existence property, that is, the terminal object of $\mathcal{L}^B$ is projective.

PROOF. The argument is inspired by Hilbert-Bernays [1934, 1939].

First note that, from a type theory $\mathcal{L}$ and a given type A, we can form a new type theory $\mathcal{L}(x)$, where x is a variable of type A. Just count all open terms of $\mathcal{L}$ which contain no free variables except x as closed terms of $\mathcal{L}(x)$.

We remark that $\mathcal{L}(x)$ is a conservative extension of $\mathcal{L}$ if and only if $\vdash \exists_{x\in A}\top$ in $\mathcal{L}$. For clearly $\vdash \exists_{x\in A}(x = x)$ in $\mathcal{L}(x)$ and, for any closed formula p of $\mathcal{L}$, if $\vdash p$ in $\mathcal{L}(x)$, then $x = x \vdash_{\{x\}} p$, hence $\exists_{x\in A}(x = x) \vdash p$ in $\mathcal{L}$.

If $\mathcal{L}$ is the internal language of a topos $\mathcal{T}$, $T(\mathcal{L}(x))$ is equivalent to the so-called "slice" topos $\mathcal{T}/A$, whose objects are arrows $B \to A$, B being any object of $\mathcal{T}$. Similarly we can form $\mathcal{L}(X)$, where X is any set of variables.

Given a type A in $\mathcal{L}$, let X_A be a set of variables containing exactly one variable x_α of type A for each closed term α of type PA such that $\vdash \exists_{x \in A} x \in \alpha$ in $\mathcal{L}$. Let F_A be the filter generated in $\mathcal{L}(X_A)$ by all formulas $x_\alpha \in \alpha$. Let X be the union of all X_A and F the filter generated by the union of all F_A, A ranging over all types of $\mathcal{L}$. Put $B\mathcal{L} \equiv \mathcal{L}(X)/F$, B for "Bernays" We claim that $B\mathcal{L}$ is a conservative extension of $\mathcal{L}$.

For suppose p is a closed formula of $\mathcal{L}$ so that $\vdash p$ in $B\mathcal{L}$. Then $p \in F$ in $\mathcal{L}(X)$, hence

$$x_{\alpha_1} \in \alpha_1, \ldots, x_{\alpha_n} \in \alpha_n \ \vdash_{\{x_{\alpha_1}, \ldots, x_{\alpha_n}\}} \ p.$$

Therefore, by existential specification,

$$\exists_{x \in A_1} x \in \alpha_1, \ldots, \exists_{x \in A_n} x \in \alpha_n \vdash p,$$

hence $\vdash p$ in $\mathcal{L}$.

Now consider the ascending chain

$$\mathcal{L} \subseteq B\mathcal{L} \subseteq \ B^2 \ \mathcal{L} \subseteq \ldots\ldots$$

and form $B^\infty \mathcal{L} \equiv \bigcup_{n \in \mathbf{N}} B^n \mathcal{L}$. Then $B^\infty \mathcal{L}$ is a conservative extension of $\mathcal{L}$ and it has the existence property. For, if $\vdash \exists_{x \in A} x \in \alpha$ in $B^\infty \mathcal{L}$, then this is so in $B^n \mathcal{L}$ for some n, hence $\vdash x_\alpha \in \alpha$ in $B^{n+1} \mathcal{L}$ and therefore in $B^\infty \mathcal{L}$. The proof is now complete.

We shall call $\mathcal{L}^B \equiv B^\infty \mathcal{L}$ the *Hilbert-Bernays completion* of $\mathcal{L}$, even though there are some technical differences between their construction and ours.

If $\alpha \equiv \{x \in A \mid \varphi(x)\}$ is such that $\vdash \exists_{x \in A} \varphi(x)$, Hilbert and Bernays stipulate $\vdash \varphi(a)$ for $a \equiv \eta_{x \in A} \varphi(x)$, where $\eta_{x \in A}$ is something like an indefinite article. On the face of it, $\eta_{x \in A} \varphi(x)$ resembles our x_α. But there is a difference: suppose α is a closed term in $B^n \mathcal{L}$, hence also in $B^{n+1} \mathcal{L}$, then there is no reason why the variable x_α^n which has been adjoined to $B^n \mathcal{L}$ should be the same as the variable x_α^{n+1} which is adjoined to $B^{n+1} \mathcal{L}$. Of course, one could modify our construction to ensure that $x_\alpha^n = x_\alpha^{n+1}$, but that would complicate matters. Less serious is another difference. Hilbert and Bernays would want $\vdash \eta_{x \in A} \varphi(x) = \eta_{x \in A} \psi(x)$ whenever $\vdash \forall_{x \in A} (\varphi(x) \Leftrightarrow \psi(x))$. This could be achieved quite easily in our setup by saying that x_α depends not on the closed term α but on its equivalence class modulo provable equality.

COROLLARY 3.2. Every non-constructive type theory $\mathcal{L}$ has a conservative extension $\mathcal{L}^B$ which satisfies Hilbert's Rule. Therefore, for each prime filter P of $\mathcal{L}^B$, the topos $T(\mathcal{L}^B/P)$ is a model topos, and thus $T(\mathcal{L}^B)$ is the topos of continuous sections of a sheaf of model toposes.

This corollary yields an easy proof of completeness for non- constructive type theories, hence for classical type theories.

It turns out that a similar result holds without the hypthesis (NC). Only then P will not range over all prime filters of the extension, but only over certain "saturated" ones.

4. Completeness of higher order intuitionistic logic.

The completeness theorem for higher order intuitionistic logic has been established before, see Lambek and Scott [1986, II§17], where a method due to Henkin [1949, 1950], as elaborated for intuistionistic first order logic by Aczel [1969], was employed. A somewhat simpler version of this proof will be presented here, before we turn to the sheaf representation, which is our ultimate objective.

A prime filter P in a type theory $\mathcal{L}$ will be called *saturated* if, whenever $\exists_{x\in A}\varphi(x)$ is in P and $\vdash \exists_{x\in A}\top$, then $\varphi(a)$ is in P for some closed term a of type A. We shall say that $\mathcal{L}$ has *enough* saturated prime filters if the intersection of all saturated prime filters is the set of closed formulas provable in $\mathcal{L}$.

PROPOSITION 4.1. (a) If P is a saturated prime filter in $\mathcal{L}$, then $\mathcal{L} \to \mathcal{L}/P \to LT(\mathcal{L}/P)$ is an interpretation of $\mathcal{L}$ in a model topos.

(b) If $\mathcal{L}$ has enough saturated prime filters, then a closed formula in $\mathcal{L}$ is provable if and only if it holds under every interpretation in a model topos $T(\mathcal{L}/P)$.

PROOF. (a) By assumption, $\mathcal{L}/P$ has the existence property for all types A such that $\vdash \exists_{x\in A}\top$, in particular when $A \equiv PB$. By Lemma 2.2, $LT(\mathcal{L}/P)$ has the existence property (EP) for all types whatsoever. Moreover, consistency (C) and the disjunction property (DP) for $LT(\mathcal{L}/P)$ follow immediately from the same properties in $\mathcal{L}/P$, as is easily seen [loc. cit. II§19]. Thus $T(\mathcal{L}/P)$ is a model topos.

(b) Suppose the closed formula q is not provable, then we can find a saturated prime filter P such that q is not in P, hence q is not provable in $\mathcal{L}/P$. Since $\mathcal{L}/P \to LT(\mathcal{L}/P)$ is a conservative extension, q does not hold in $T(\mathcal{L}/P)$.

Having established completeness for type theories with enough saturated prime filters, we shall now show that every type theory has a conservative extension with enough saturated prime filters, from which its completeness follows.

We begin by constructing the Henkin completion $\mathcal{L}^H$ of a type theory $\mathcal{L}$. Recall from the proof of Proposition 3.1 that, if x is a variable of type A in $\mathcal{L}$, $\mathcal{L}(x)$ is a conservation extension of $\mathcal{L}$ if and only if $\vdash \exists_{x\in A}\top$ in $\mathcal{L}$. Assume that this is the case and let X_A be a set of variables containing exactly one variable x_α for each closed term α of type PA. (If we wish, we may identify x_α with $x_{\alpha'}$ when $\vdash \alpha = \alpha'$ in $\mathcal{L}$, but it is crucial that x_α differs from $x_{\alpha'}$ when this is not the case.) Let X be the union of all X_A for which $\vdash \exists_{x\in A}\top$ and put $H\mathcal{L} \equiv \mathcal{L}(X)$. This is a conservative extension of $\mathcal{L}$. Now consider the ascending chain

$$\mathcal{L} \subseteq H\mathcal{L} \subseteq H^2\mathcal{L} \subseteq \ldots\ldots$$

and form $\mathcal{L}^H \equiv H^\infty\mathcal{L} \equiv \bigcup_{n\in\mathbf{N}} H^n\mathcal{L}$. This is also a conservative extension, which we shall call the *Henkin completion* of $\mathcal{L}$.

Except for the labelling of variables, which serves a technical purpose in the proof of the lemma below, the Henkin completion $\mathcal{L}^H$ is just $\mathcal{L}(V)$, where V is a set of variables containing sufficiently many variables of each type A such that $\vdash \exists_{x\in A}\top$.

Henkin's original idea [1949] is contained in the proof of the following lemma, which we shall make much use of in this and the next section. (Freyd [1972] uses a similar construction to show that every small Boolean topos can be logically embedded in a product of well-pointed topoi.)

LEMMA 4.2. Suppose F_0 is a filter in the Heyting algebra of closed formulas of $\mathcal{L} \equiv H^0\mathcal{L}$ and K_0 is an ideal (i.e. dual filter) such that $F_0 \cap K_0 = \phi$. Then there exists a saturated prime filter P in $H^\infty\mathcal{L}$ such that $F_0 \subseteq P$ and $P \cap K_0 = \phi$.

PROOF. By Zorn's Lemma, we may pick a filter P_0 in $H^0\mathcal{L}$ containing F_0 maximal with the property that $P_0 \cap K_0 = \phi$. It is well-known and easily shown that P_0 is a prime filter. (Incidentally, we could also have obtained a prime filter containing F_0 and not meeting K_0 as the complement of a maximal ideal Q_0 containing K_0 such that $F_0 \cap K_0 = \phi$.)

Let F_1 be the filter in $H^1\mathcal{L}$ generated by P_0 and all $x_\alpha \in \alpha$ for which $\exists_{x\in A} x \in \alpha$ is in P_0 and let K_1 be the ideal in $H^1\mathcal{L}$ generated by K_0. Then $F_1 \cap K_1 = \phi$. For suppose that $q_1 \in F_1 \cap K_1$, then $q_1 \vdash q_0$ for some $q_0 \in K_0$, hence $q_0 \in F_1 \cap K_0$. Therefore, there is a $p_0 \in P_0$ such that $p_0,\ x_{\alpha_1} \in \alpha_1, \ldots, x_{\alpha_n} \in \alpha_n \vdash_{\{x_{\alpha_1},\ldots,x_{\alpha_n}\}} q_0$, and so, by existential specification,

$$p_0,\ \exists_{x\in A_1} x \in \alpha_1, \ldots, \exists_{x\in A_n} x \in \alpha_n \vdash q_0,$$

so that $q_0 \in P_0 \cap K_0$, a contradiction.

Now repeat the above argument to produce a prime filter P_1 in $H^1\mathcal{L}$ containing F_1 such that $P_1 \cap K_1 = \phi$ and continue in the same way. Since a closed term α of type PA in $H^n\mathcal{L}$ is also in $H^{n+1}\mathcal{L}$, supposing $\exists_{x\in A} x \in \alpha$ is in P_n, we should be careful to distinguish the variable x_α^n in $H^{n+1}\mathcal{L}$ from the variable x_α^{n+1} in $H^{n+2}\mathcal{L}$. In particular, the x_α above should really be x_α^0.

In this manner one obtains filters F_n and prime filters P_n such that

$$F_0 \subseteq P_0 \subseteq F_1 \subseteq P_1 \subseteq F_2 \subseteq P_2 \subseteq$$

Let P be the union of the P_n. Then clearly P contains F_0 and $P \cap K_0 = \phi$. To see that P is saturated, suppose $\exists_{x\in A} x \in \alpha$ is in P, then it is is some P_n, so $x_\alpha^n \in \alpha$ is in P_{n+1}, hence in P.

PROPOSITION 4.3. (Completeness of intuitionistic higher order logic.) Every type theory $\mathcal{L}$ has a conservative extension $\mathcal{L}^H$, its Henkin completion, obtained from $\mathcal{L}$ by adjoining sufficiently many variables, which has enough saturated prime filters. Thus, a closed formula of $\mathcal{L}$ is provable if and only if it holds in every model topos $T(\mathcal{L}^H/P)$, P being any saturated prime filter of $\mathcal{L}^H$.

PROOF. It suffices to show that $\mathcal{L}^H$ has enough saturated prime filters P, since then $T(\mathcal{L}^H/P)$ will be a model topos by Proposition 4.1. Let q be any closed formula of $\mathcal{L}^H$ not provable in $\mathcal{L}^H$. We may assume that q is in $\mathcal{L}' \equiv H^n\mathcal{L}$. Apply Lemma 4.2 to $\mathcal{L}'$ and note that $H^\infty\mathcal{L}' = H^\infty\mathcal{L} = \mathcal{L}^H$. Let F_0' be the filter generated by $\top$ in $\mathcal{L}'$ and K_0' the ideal generated by q in $\mathcal{L}'$. Then the lemma produces a saturated prime filter P of $\mathcal{L}^H$ not containing q.

What is the relation between the Hilbert-Bernays completion $\mathcal{L}^B$ studied in Section 3 and the Henkin completion $\mathcal{L}^H$ of Section 4? The former adjoins a witness to every existential statement which can be proved. The latter adjoins a potential witness to every existential statement, provable or not, provided this can be done conservatively.

To see that $\mathcal{L}^H$ is indeed different from $\mathcal{L}^B$, consider the closed term $\omega \equiv \{x \in \Omega \mid x\}$ of type $P\Omega$. Then $\vdash \exists_{x\in\Omega} x \in \omega$ but not $\vdash x_\omega \in \omega$ (since otherwise $\vdash \forall_{x\in\Omega} x$) unless $\mathcal{L}$ is inconsistent, even though $x_\omega \in \omega$ is in all saturated prime filters containing F_1.

5. Sheaf representation of the Henkin completion.

We shall now establish our principal result, which strengthens the completeness theorem of the previous section.

THEOREM 5.1. Every type theory $\mathcal{L}$ has a conservative extension $\mathcal{L}^H$, its Henkin completion, obtained from $\mathcal{L}$ by adjoining sufficiently many variables, such that the topos generated by $\mathcal{L}^H$ is the topos of continuous global sections of a sheaf of model toposes.

PROOF. We begin by constructing a new topological space, the ***saturated spectrum*** of $\mathcal{L}$, also denoted *Spec* $\mathcal{L}$: its points are the saturated prime filters P of $\mathcal{L}^H$ and its basic open sets have the form

$$V(p) \equiv \{P \in \ Spec\, \mathcal{L} \mid p \in P\},$$

where p is any closed formula of $\mathcal{L}$. Here are some more or less obvious properties of $V(p)$:

(1) $V(p \wedge q) = V(p) \cap V(q)$;
(2) $V(p \vee q) = V(p) \cup V(q)$;
(3) $V(p) \subseteq V(q)$ if and only if $p \vdash q$;
(4) $\bigcap V(p) = (p)$, the principal filter in $H^\infty \mathcal{L}$;
(5) $V(p)$ is compact.

(1) holds because the points of the spectrum are filters; (2) holds because they are prime. To show (3) in the less obvious direction, suppose $V(p) \subseteq V(q)$ but not $p \vdash q$. We obtain a contradiction by finding a saturated prime filter P in $\mathcal{L}^H$ such that $p \in P$ but not $q \in P$. This is done by Lemma 4.2, taking F_0 to be the filter in $\mathcal{L}$ generated by p and K_0 the ideal in $\mathcal{L}$ generated by q. (4) is proved like Proposition 4.3 with p in place of $\top$

Another application of Lemma 4.2 yields (5). For assume that

$$V(p) \subseteq \bigcup_{i \in I} V(q_i),$$

I being some set of indices. We claim that p belongs to the ideal K_0 in $\mathcal{L}$ generated by the q_i. Otherwise let F_0 be the filter generated by p in $\mathcal{L}$, then $F_0 \cap K_0 = \phi$, so we could find a saturated prime filter P containing F_0 but not meeting K_0, hence containing p but none of the q_i, thus contradicting our assumption. It follows that I has a finite subset $\{i_1, \ldots, i_n\}$ such that $p \vdash q_{i_1} \vee \quad \vee q_{i_n}$. From (2) and (3) we then infer that

$$V(p) \subseteq V(q_{i_1}) \cup \ldots \cup V(q_{i_n}),$$

which shows that $V(p)$ is compact.

We are now ready to construct a presheaf. If p is a closed formula in $\mathcal{L}$, one can see from (3) or from (4) that $V(p)$ determines the principal filter (p) in $\mathcal{L}^H$. With each basic open set $V(p)$ we associate the topos $\mathcal{T}(p) \equiv \mathcal{T}(\mathcal{L}^H/(p))$. Whenever $V(p) \subseteq V(q)$, one infers from (3) that $(p) \subseteq (q)$, hence one obtains a canonical translation $\mathcal{L}^H/(q) \to \mathcal{L}^H/(p)$, and consequently a logical functor $\mathcal{T}(q) \to \mathcal{T}(p)$. It is well-known how to extend this assignment $V(p) \mapsto \mathcal{T}(p)$ to a contravariant functor from the spectrum of $\mathcal{L}^H/\mathcal{L}$, viewed as a category, to the meta-category of toposes: to a union of basic open sets one assigns the inverse limit of the corresponding toposes.

We claim that the presheaf we have just constructed is in fact a sheaf. In view of compactness, it suffices to check that, whenever

$$V(p) = V(q_1) \cup \ldots \cup V(q_n),$$

the following is an equalizer diagram:

$$\mathcal{T}(p) \to \prod_{i=1}^{n} \mathcal{T}(q_i) \rightrightarrows \prod_{i,j=1}^{n} \mathcal{T}(q_i \wedge q_j).$$

Note that all the toposes $\mathcal{T}(p)$, $\mathcal{T}(q_i)$ and $\mathcal{T}(q_i \wedge q_j)$ have as objects equivalence classes of closed terms of type PA for any type A in $\mathcal{L}^H$, but the equivalence classes are defined modulo p, modulo q_i and modulo $q_i \wedge q_j$ respectively.

Suppose the α_i modulo q_i $(i = 1, \ldots, n)$ are objects of $\mathcal{T}(q_i)$ so that the restrictions of α_i modulo q_i and α_j modulo q_j to $\mathcal{T}(q_i \wedge q_j)$ coincide, that is,

(i) $q_i \wedge q_j \vdash \alpha_i = \alpha_j$,

for all i and $j = 1, \ldots, n$. We wish to find a unique object α modulo p in $\mathcal{T}(p)$ which restricts to α_i modulo q_i in $\mathcal{T}(q_i)$. Note that by (2) and (3) above, we also have

(ii) $p \vdash q_1 \vee \quad \vee q_n$,

(iii) $q_i \vdash p \quad (i = 1, \ldots, n)$.

The following statement, known as ***definition by cases***, is easily shown to hold in any intuitionistic type theory, in particular in $\mathcal{L}^H$:

$$\bigwedge_{i,j=1}^{n} ((q_i \wedge q_j) \Rightarrow \alpha_i = \alpha_j),\ \bigvee_{i=1}^{n} q_i \vdash \exists!_{u \in PA} \bigwedge_{i=1}^{n} (q_i \Rightarrow u = \alpha_i).$$

In view of (i) and (ii), we obtain in $\mathcal{L}^H$:

$$p \vdash \exists!_{u \in PA} \bigwedge_{i=1}^{n} (q_i \Rightarrow u = \alpha_i).$$

Passing now to $LT(\mathcal{L}^H/(p))$, the internal language of $\mathcal{T}(p)$, we have

$$\vdash \exists!_{u \in \underline{PA}} \bigwedge_{i=1}^{n} (\underline{q}_i \Rightarrow u = \underline{\alpha}_i).$$

By Lemma 2.3, there is a unique arrow $\underline{\alpha} : \underline{1} \to \underline{PA}$ in $\mathcal{T}(p)$ such

$$\vdash \bigwedge_{i=1}^{n} (\underline{q_i} \Rightarrow \underline{\alpha} = \underline{\alpha_i}).$$

In view of the properties of the translation $\eta_{\mathcal{L}^H/(p)}$ discussed at the end of Section 1, $\underline{\alpha}$ is the image of a closed term α of type PA in $\mathcal{L}^H/(p)$, unique up to provable equality, and

$$\vdash \bigwedge_{i=1}^{n} (q_i \Rightarrow \alpha = \alpha_i)$$

in $\mathcal{L}^H/(p)$, that is, in $\mathcal{L}^H$

$$p \vdash \bigwedge_{i=1}^{n} (q_i \Rightarrow \alpha = \alpha_i).$$

It then follows from (iii) that

$$q_i \vdash \alpha = \alpha_i \qquad (i = 1, \ldots, n),$$

hence from (ii) that

$$p \vdash \alpha = \alpha_i \qquad (i = 1, \ldots, n).$$

Thus α modulo p is the unique object of $\mathcal{T}/(p)$ which restricts to the α_i modulo q_i in $\mathcal{T}/(q_i)$. We may therefore write α in place of α_i.

As far as objects are concerned, the diagram under consideration is thus an equalizer diagram. Let us now consider arrows. An arrow $f_i : \alpha \to \beta$ in $\mathcal{T}(q_i)$, with β of type PB say, is given by its graph ρ_i, a closed term of type $P(A \times B)$ in $\mathcal{L}^H/(q_i)$, such that

$$q_i \vdash \rho_i \in Fcn(\alpha, \beta)$$

in $\mathcal{L}^H$, where $Fcn(\alpha,\beta)$ is short for:

$$\{w \in P(A\times B) \mid w \subseteq \alpha\times\beta \wedge \forall_{x\in A}(x\in\alpha \Rightarrow \exists!_{y\in B} <x,y> \in w)\}.$$

We wish to find a unique arrow $f \quad \alpha \to \beta$ in $\mathcal{T}(p)$ which restricts to f_i in $\mathcal{T}(q_i)$.

As above, we find a closed term ρ of type $P(A\times B)$ in $\mathcal{L}^H/(p)$, unique up to provable equality, such that

$$q_i \vdash \rho = \rho_i \qquad (i = 1,\ldots,n).$$

It follows that

$$q_i \vdash \rho \in Fcn(\alpha,\beta) \qquad (i = 1,\ldots,n),$$

hence by (ii) that

$$p \vdash \rho \in Fcn\ (\alpha,\beta).$$

Therefore ρ is the graph of an arrow $f \quad \alpha \to \beta$ in $\mathcal{T}(p)$. Thus the diagram in question is an equalizer diagram also as far as arrows are concerned.

We have thus shown that the assignment of the topos $\mathcal{T}(p) \equiv T(\mathcal{L}^H/(p))$ to the basic open set $V(p)$ yields a sheaf. It remains to determine the stalks of this sheaf. The stalk at P is defined to be the direct limit of all toposes $\mathcal{T}(p)$ such that $P \in V(p)$, that is, $p \in P$. This direct limit is to be calculated in the meta-category of toposes and logical morphisms. Since the functor T is left adjoint to the functor L, it suffices to find the direct limit of the $\mathcal{L}^H/(p)$ in the meta-category of type theories and translations. As all these languages have the same types and terms, we need only take the union of the principal filters (p), thus obtaining the filter P. It follows that the stalk at P is $T(\mathcal{L}^H/P)$, which was shown to be a model topos in Proposition 4.1.

The proof of the theorem is now complete.

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