Introduction to multicategories in logic and computer science

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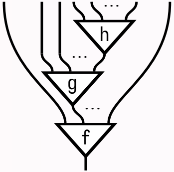
Introduction

- Many concepts in logic and computer science involve “operations” of several inputs, and these “operations” can moreover be combined
- Multicategories are a simple algebraic gadget that capture this pattern



$f(x_1, \ldots, g(y_1, \ldots, h(z_1, \ldots, z_n)), x_m)$

- By making common structure explicit, we can put diverse notions on the same footing and thus compare, abstract, learn, and teach
- Semantics, but close to syntax: interface between logic and algebra
- This talk (mostly) assumes no specific background

Outline

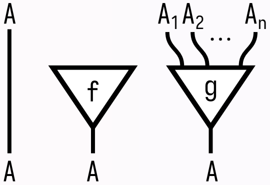
1. Definition + initial examples, e.g. functions, logic, categories, incomplete processes
2. Morphisms of multicategories
3. Free multicategories and cut elimination
4. Symmetric multicategories
5. Cartesian multicategories
6. Algebraic structures via multicategories
7. Representable multicategories

Definition: multicategory

A *multicategory* $\mathcal{M}$ is a **multigraph** with **units** and **composition**. A *multigraph* comprises,

- a set of objects $\{A, A_1, \dots\}$, and
- for each list of objects $A_1, \dots, A_n$ and object A, a set $\mathcal{M}(A_1, \dots, A_n; A)$,

Units or *identities* are distinguished elements 1_A in $\mathcal{M}(A; A)$,



$$1_A \in \mathcal{M}(A; A) \quad f \in \mathcal{M}(; A) \quad g \in \mathcal{M}(A_1, \dots, A_n; A)$$

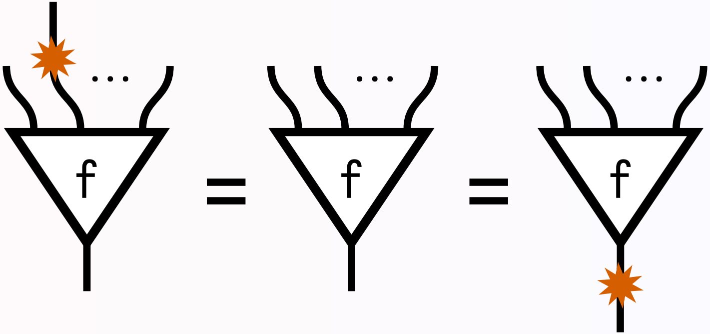
➙ In a multigraph we usually call the elements of the sets $\mathcal{M}(A_1, \dots, A_n; A)$ *operations* or *generators*, and if we have a multicategory we usually call them *(multi)morphisms*.

Definition: multicategory (cont.)

Composition is given by functions,

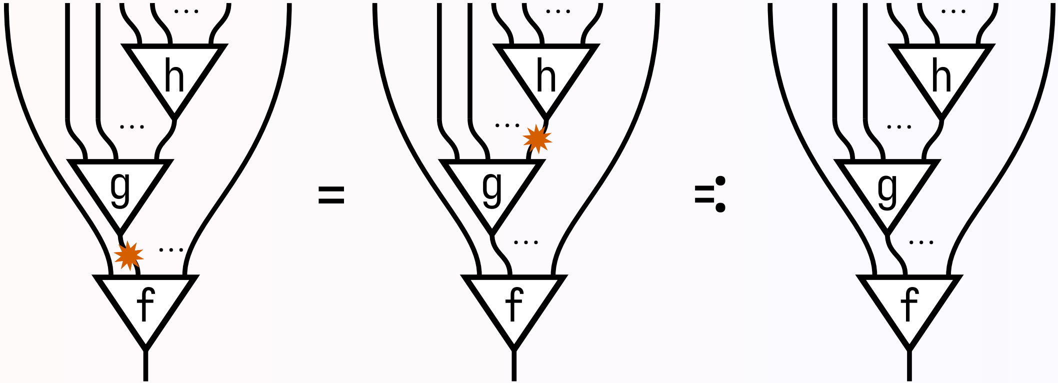
$$\mathbin{⨟}^{X_i, Z_j}_{Y} : \mathcal{M}(X_1, ..., X_n; Y) \times \mathcal{M}(Z_1, ..., Z_i, Y, Z_{i+1}, ...Z_m; Z) \to \mathcal{M}(Z_1, ..., Z_i, X_1, ..., X_n, Z_{i+1}, ..., Z_m; Z)$$

satsifying laws: composition is *unital* ...



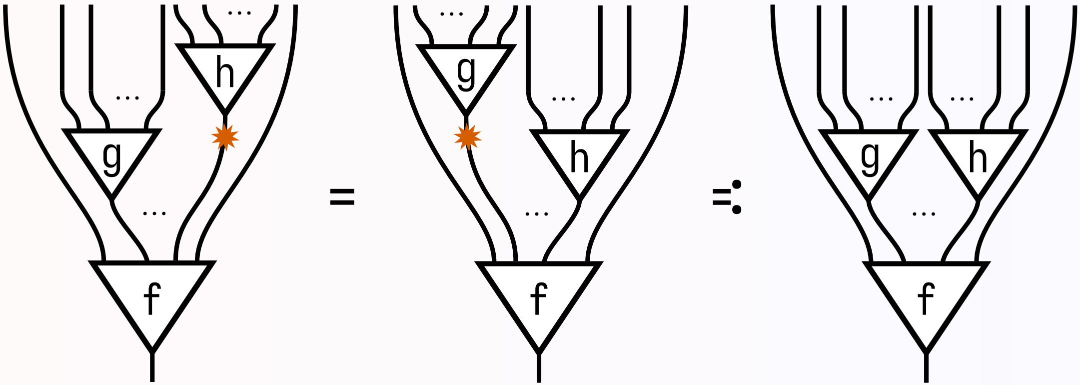
Definition: multicategory (cont.)

… composition is *associative*,



Definition: multicategory (cont.)

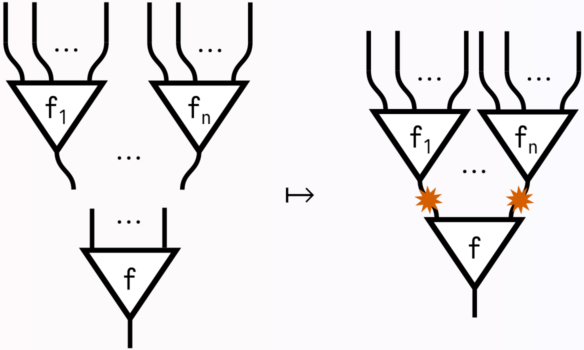
and composition satisfies *interchange*.



➻ Multicategories are sometimes called *(coloured, planar) operads*.

Alternative composition structure

We could give composition in terms of combining *tuples of multimorphisms* into a multimorphism,



... which we ask to satisfy appropriate unitality and associativity laws.

Using unitality/interchange, we see that these forms of compositions are interdefinable.

Example: sets and multi-ary functions

The multicategory **Set** has,

- objects: sets,
- multimorphisms: multi-ary functions $f : A_1, \ldots, A_n \to A$,
- units: identity functions,
- composition: substitution, e.g. for

$$f : A_1, \ldots, A_n \to A \qquad g : B_1, \ldots, A, \ldots, B_m \to C$$

$$\begin{aligned}(f \mathbin{⨟}_A g) &: B_1, \ldots, A_1, \ldots, A_n, \ldots, B_m \to C \\ &(b_1, \ldots, a_1, \ldots, a_n, \ldots, b_m) \mapsto g(b_1, \ldots, f(a_1, \ldots, a_n), \ldots, b_m)\end{aligned}$$

Similarly: vector spaces and multilinear maps.

Example: (a fragment of) propositional logic

- objects: propositions $A, B ::= P \in \mathcal{P} \mid \top \mid A \wedge B$
- morphisms: $A_1, ..., A_n \to A$ entailments, inductively generated by

$$\frac{}{\Gamma \vdash \top} \qquad \frac{A \in \mathcal{P}}{A \vdash A} \qquad \overset{\textsc{cut}}{\frac{\Gamma \vdash A \qquad \Delta, A, \Delta' \vdash B}{\Delta, \Gamma, \Delta' \vdash B}} \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C}$$

- identities: the entailments $A \vdash A$, composition: the CUT rule

➻ When thinking of a multimorphism as an entailment, we usually write $\vdash$ instead of $\to$.

This does not yet capture all the structure we might expect, e.g.

- if we have $A, B \vdash C$, we should also have $B, A \vdash C$ and vice-versa.

Feature not a bug: substructural logic

The following rules, which we do not have in multicategories, are called structural rules:

- Exchange/symmetry: $A, B \vdash C \leftrightarrow B, A \vdash C$
- Contraction: $A, A, B \vdash C \rightarrow A, B \vdash C$
- Weakening: $A \vdash B \rightarrow A, C \vdash B$

Later we will add this structure, but logic without it is interesting and necessary!

- Lambek/Chomsky grammars: word order matters
- *Linear logic* is a logic of "resources": two "assumptions" of a resource are different from one

Lambek, 1958 – *The Mathematics of Sentence Structure*

Girard, 1987 – *Linear Logic*

A class of examples: categories

Categories are multicategories having only unary multimorphisms, i.e. $f : A \to B$.

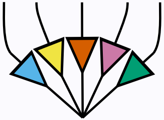
Every multicategory $\mathcal{M}$ has an underlying category $\mathcal{M}_{-}$: take only unary morphisms.

Taking underlying categories is *right adjoint* to the regarding a category as a unary multicategory,

$$\frac{\mathbb{C}^1 \to \mathcal{M}}{\mathbb{C} \to \mathcal{M}_{-}}$$

Every category $\mathbb{C}$ also (freely) gives rise to a *sequential/cocartesian* multicategory,

$$\mathbb{C}^{\text{cocart}}(A_1, \ldots, A_n; A) := \mathbb{C}(A_1; A) \times \ldots \times \mathbb{C}(A_n; A)$$

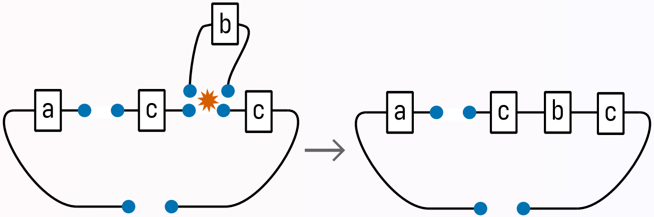


Pisani, 2014 – *Sequential Multicategories*

Example: spliced words

Let Σ^* be the monoid of words over Σ. The multicategory $S(\Sigma^*)$ of *spliced words* has:

- one object, denoted • •
- n-ary morphisms $S(\Sigma^*)(\bullet\ \bullet, \ldots, \bullet\ \bullet; \bullet\ \bullet)$ are words with n gaps ($n+1$ tuples of words):
- identities are the gapped word $(\varepsilon, \varepsilon)$
- composition is defined via concatenation

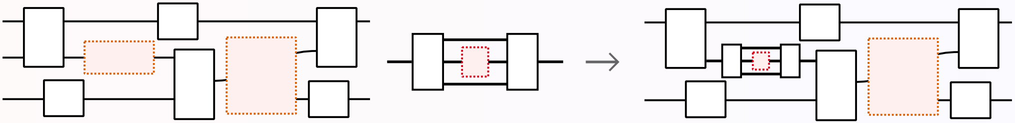


➛ This construction works more generally for a *category*.

Melliès and Zeilberger, 2025 – *Categorical contours...*

Example: process contexts

More generally for a *monoidal category*, there is a multicategory of contexts,



➻ *cf.* "little disks", "little cubes" and "swiss cheese" multicategories in topology.

Spivak, 2013 – *The Operad of Wiring Diagrams*

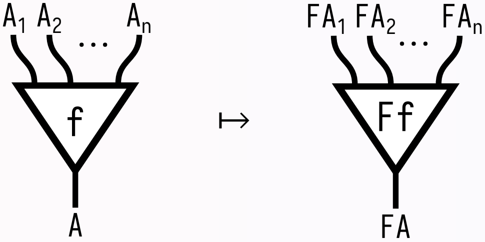
E and Román, 2025 – *Context-free Languages of String Diagrams*

E, Hefford, and Román, 2024 – *The Produoidal Algebra of Process Decomposition*

Morphisms between multigraphs and multicategories

A *morphism of multigraphs* $F : M \to N$ is a function F between the objects and functions,

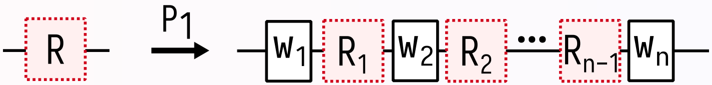
$$F_{(A_1,\ldots,A_n;A)} : M(A_1, \ldots, A_n; A) \to N(FA_1, \ldots, FA_n; FA).$$

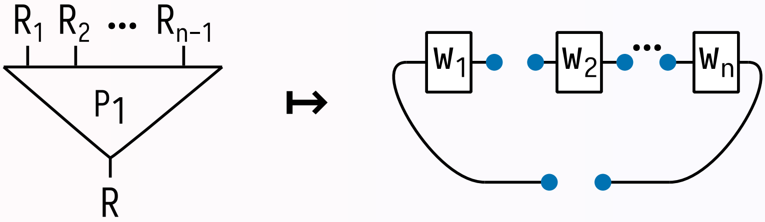


A *morphism of multicategories* (or *multifunctor*) is a morphism of underlying multigraphs, preserving identities and composites,

$$F(1_A) = 1_{FA}$$
$$F(f \mathbin{⨟}_A g) = F(f) \mathbin{⨟}_{FA} F(g)$$

Example: Context-free grammars as morphisms of multigraphs



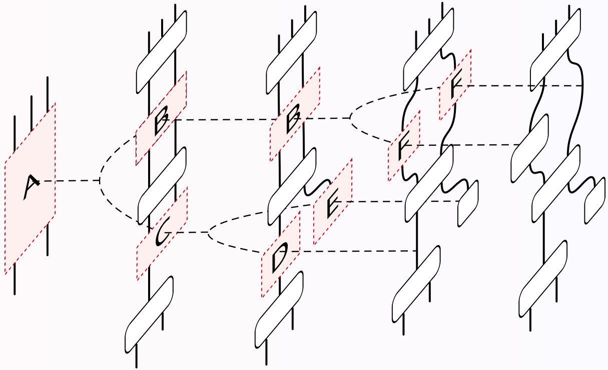


$$M \longrightarrow |S(\Sigma^*)|$$

Melliès and Zeilberger, 2025 – *Categorical contours...*

Example: Context-free grammars over string diagrams

The same idea but using contexts in string diagrams instead of spliced words.

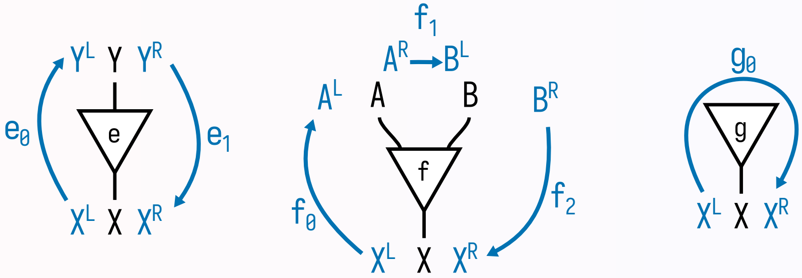


Examples include context-free grammars of trees and hypergraphs.

E and Román, 2025 – *Context-free languages of string diagrams*

Contour: canonical linearization of a multicategory

Every multicategory M gives rise to a category called its *contour*:



Taking contours is *left adjoint* to taking *splices*, meaning we have the rule of inference:

$$\frac{\mathcal{C}M \to G}{M \to \mathcal{S}G}$$

Melliès and Zeilberger, 2025 – *Categorical contours...* E, Hefford, and Román, 2024 – *Produoidal algebra...*

A class of examples: free multicategories

The *free multicategory* over a multigraph M is a multicategory $\mathcal{F}M$ such that for any multicategory N, morphisms of multicategories,

$$\mathcal{F}M \to N$$

are in (natural) bijection with morphisms of multigraphs,

$$M \to |N|.$$

➥ $|N|$ denotes the underlying multigraph of the multicategory N.

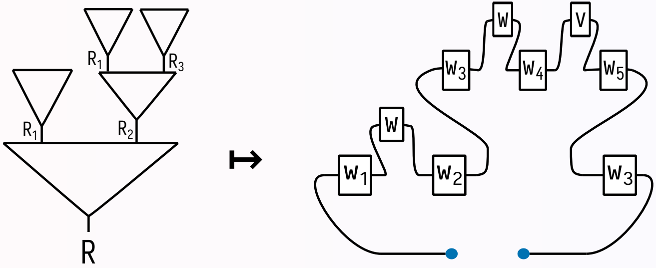
We say "the" ... "a" because this definition specifies $\mathcal{F}M$ up to unique isomorphism.

Context-free derivations via morphisms of multicategories

If we think of a multigraph M as giving typing judgments in a CFG, roughly speaking $\mathcal{F}M$ has multimorphisms all the (partial) *derivations* or (open) *trees* built from M.

From a grammar to derivations...

$$\frac{M \to |\mathcal{S}(\Sigma^*)|}{\mathcal{F}M \to \mathcal{S}(\Sigma^*)}$$



Melliès and Zeilberger, 2025 – *Categorical contours...*

A tautological construction of the free multicategory

Let Γ, Δ denote lists of objects, and G be a *multigraph*.

Define a multicategory with objects those of G and morphisms inductively defined by

$$\frac{A \in G}{\mathrm{id}_A : (A \vdash A)}\ \textsc{id} \qquad \frac{f \in G(\Gamma; A)}{f : (\Gamma \vdash A)}\ \textsc{gen} \qquad \frac{g : (\Delta \vdash A) \qquad h : (\Gamma, A, \Gamma' \vdash B)}{g \mathbin{;}_A h : (\Gamma, \Delta, \Gamma' \vdash B)}\ \textsc{cut}$$

quotiented by the least congruence for term constructors generated by

$$f = \mathrm{id} \mathbin{;} f = f \mathbin{;} \mathrm{id}$$
$$(f \mathbin{;} g) \mathbin{;} h = f \mathbin{;} (g \mathbin{;} h)$$
$$h \mathbin{;}_B (g \mathbin{;}_A f) = g \mathbin{;}_A (h \mathbin{;}_B f)$$

This satisfies the universal property of the free multicategory.

Free multicategories and cut elimination

This tautological construction is unsatisfactory for some purposes.

a) We have to reason up to the equivalence relation

b) The presence of CUT as a primitive *rule* in a logic stymies proof search.

$$\frac{?_1 : (\Delta \vdash A) \qquad ?_2 : (\Lambda, A, \Lambda' \vdash B)}{? : (\Gamma \vdash B)}$$

Potentially infinite number of lemmas A...

This is one reason logicians are interested in *cut elimination*.

We can use multicategories to prove cut admissibility/elimination.

Lambek, 1969 – *Deductive Systems and Categories II*

Shulman, 2016 – *Categorical logic from a categorical point of view*

Free multicategories and cut admissibility

We can make cut invisible by introducing formal variables in our contexts/terms.

$$\frac{(A \in G)}{x : A \vdash x : A} \qquad \frac{f \in G(A_1, \ldots, A_n; B) \qquad \Gamma_1 \vdash t_1 : A_1 \qquad \ldots \qquad \Gamma_n \vdash t_1 : A_n}{\Gamma_1, \ldots, \Gamma_n \vdash f(t_1, \ldots, t_n) : B}$$

➙ Variables in a context are distinct. They appear *exactly once* in each term, *in order* (“*linearly*”).

Define a multigraph having objects those of G and multimorphisms $A_1, \ldots, A_n \to A$ the derivable *terms-in-context* $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$.

Composition is *substitution*, defined recursively on the term-in-context $\Gamma, x : A, \Gamma' \vdash s : B$ into which we substitute:

$$s[x := t] := \begin{cases} t & \text{if } s = x \\ f(s_1, \ldots, s_i[x := t], \ldots, s_n) & \text{if } s = f(s_1, \ldots, s_n) \end{cases}$$

By induction this is unital, associative, and satisfies interchange. No need to quotient.

Free multicategories and cut admissibility

Substitution gives us a way to construct the conclusion of the following rule given the antecedents:

$$\frac{\Gamma, x : A, \Gamma' \vdash s : B \qquad \Delta \vdash t : A}{\Gamma, \Delta, \Gamma' \vdash s[x := t] : B}$$

This is *cut admissibility*: using the above rule ($\textsc{cut}$) does not let us derive more than we can with only two rules.

By the universal property, we have an *isomorphism* $\mathcal{F}G_{\text{nocut}} \cong \mathcal{F}G_{\text{cut}}$.

The induced $\mathcal{F}G_{\text{cut}} \to \mathcal{F}G_{\text{nocut}}$ is essentially a *normalization* procedure,

$$[(a \fatsemi b) \fatsemi ((c \fatsemi d) \fatsemi e)] \mapsto a(b(c(d(e(x)))))$$

Internal language

The cut-free presentation of free multicategories is sometimes called an *internal language*: a *sound* and *complete* language for reasoning about multicategories.

Sound: any theorem we prove in the internal language of $\mathcal{F}G$ holds in any multicategory M in which we have an interpretation of the multigraph $G \to |M|$.

Complete: if a theorem holds in all multicategories, it holds in the internal language. The internal language forms a multicategory (the free one), so this follows trivially.

➜ We can work with the set-and-multiary-function-like syntax and use the universal property of $\mathcal{F}G$ to interpret statements in an arbitrary multicategory M.

Algebraic structures via multicategories

What is a morphism of multicategories $\mathcal{F}G \to \mathbf{Set}$?

Consider the multigraph G with one object and operation $\quad \text{ADD} : *, * \to *$

A morphism $G \to |\mathbf{Set}|$ is a set equipped with a binary function.

The corresponding $\mathcal{F}G \to \mathbf{Set}$ equips the set with all of the "derived operations".

We can quotient $\mathcal{F}G$ by equations between terms in the same context, e.g.

$$\frac{}{x : *, y : *, z : * \vdash \text{ADD}(x, \text{ADD}(y, z)) = \text{ADD}(\text{ADD}(x, y), z) : *}$$

Then a morphism $\mathcal{F}G/\equiv \to \mathbf{Set}$ is a precisely a semigroup.

But we cannot yet express the theory of e.g. *commutative semigroups*, since we cannot even form the term

$$x : *, \ y : * \vdash \text{ADD}(y, x) : *$$

Extra structure: symmetric multicategories

Sometimes we can naturally permute the inputs of morphisms in a multicategory.

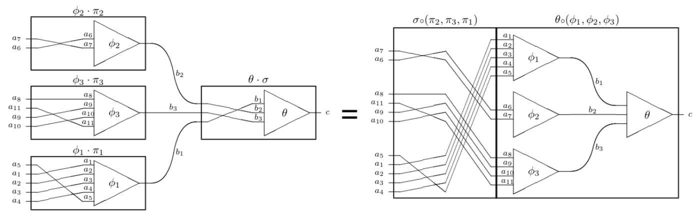
- If $f : X, Y \to Z$ is a function, we have a function $Y, X \to Z : (y, x) \mapsto f(x, y)$.
- If $P, Q \vdash R$ is an entailment in classical logic, we have an entailment $Q, P \vdash R$.

Non-examples: context-free derivations, spliced words

A *symmetric multicategory* is a multicategory M equipped with functions,

$$- \cdot \sigma : M(A_{\sigma(1)}, ...A_{\sigma(n)}; B) \to M(A_1, ..., A_n; B)$$

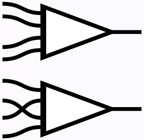
for every permutation $\sigma : n \to n$, such that $f \cdot 1 = f$, $\quad f \cdot (\sigma\sigma') = (f \cdot \sigma) \cdot \sigma'$, and



Free symmetric multicategories

We obtain an internal language for symmetric multicategories by adding the following rule to the cut-free type theory,

$$\frac{\Gamma, x : A, y : B, \Delta \vdash t : C}{\Gamma, y : B, x : A, \Delta \vdash t : C}$$



Symmetric multicategories capture both the exchange rule in logic and commutative algebraic theories, e.g. commutative semigroups.

$$\frac{x : *, y : * \vdash \text{ADD}(x, y) : *}{y : *, x : * \vdash \text{ADD}(x, y) : *}$$

This latter is α-equivalent to $x : *, y : * \vdash \text{ADD}(y, x) : *$. We can ask,

$$\frac{}{x : *, y : * \vdash \text{ADD}(x, y) = \text{ADD}(y, x) : *}$$

Extra structure: cartesian multicategories

We still cannot form terms such as $x : A \vdash f(x, x) : B$, in which the same variable appears more than once (logically: contraction), nor terms such as $x : A, y : B \vdash f(x) : C$ which do not use a variable (logically: weakening)

Cartesian multicategories are multicategories equipped with functions

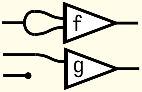
$$M(A_{f(1)}, \ldots, A_{f(n)}; A) \to M(A_1, \ldots, A_m; A)$$

where $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ is any function, which must satisfy equations generalizing those for symmetric multicategories (see nLab).

➜ Key example: sets and multi-ary functions.

e.g. given $f : A, A \to B$ we have an $f' : A \to B : a \mapsto f(a, a)$

given $g : A \to B$ we have an $g' : A, C \to B : (a, c) \mapsto g(a)$



Algebraic structures via cartesian multicategories

With cartesian multicategories, we can form terms such as
$x : A, y : A, z : A \vdash (x \times y) + (x \times z) : A$, or $x : A, y : A \vdash x : A$.

⇝ Cartesian multicategories define the class of *algebraic theories*.

We can form the free cartesian multicategory on a multigraph by adding the rules

$$\frac{\Gamma, x : A, y : A, \Gamma' \vdash f : B}{\Gamma, x : A, \Gamma' \vdash f : B} \qquad \frac{\Gamma \vdash g : B}{\Delta, x : A, \Delta' \vdash g : B}$$

Morphisms of multicategories preserving cartesian structure,

$$\mathcal{F}_{\mathrm{c}} G \to \mathbf{Set}$$

are models of the theory $\mathcal{F}_{\mathrm{c}} G$, e.g. monoids, rings, etc.

Property: Representable multicategories

➜ "an n-ary function is a function $A_1 \times \ldots \times A_n \to B$"

"in a sequent, think of the comma on the left of the turnstile as being conjunction"

"a multilinear function is equivalently a linear function from the tensor product"

These ideas are formalized by multicategories that have the property of being **representable**.

A *representation* of a list of objects $A_1, \ldots, A_n$ in a multicategory M is an object $\bigotimes_i A_i$ of M equipped with a multimorphism,

$$m_{\vec{A}} : A_1, \ldots, A_n \to \bigotimes_i A_i$$

such that composition with $m_{\vec{A}}$ is a *bijection* (natural in A),

$$M(\Gamma, \bigotimes_i A_i, \Delta \,;\, A) \xrightarrow{\cong} M(\Gamma, A_1, \ldots, A_n, \Delta \,;\, A)$$

A multicategory is *representable* if every list of objects has a representation.

Hermida, 2000 – *Representable multicategories*

Representable multicategories

Examples:

- $A_1 \times \ldots \times A_n$ in the multicategory of sets and multi-ary functions,
- $A_1 \wedge \ldots \wedge A_n$ in multicategory of propositions and entailments,
- $A_1 \otimes \ldots \otimes A_n$ in the multicategory of vector spaces and multilinear maps

➜ Proposition: representations are unique up to unique isomorphism.

That is, any two implementations of a tensor product are canonically isomorphic.

Representable multicategories are *equivalent* to *monoidal categories*, but if we model our programming language in terms of a monoidal category, we have to ask for these isomorphisms as part of the data and check *coherence*.

More multicategories and beyond

- Polycategories: many inputs and many outputs. Applications in logic,

  Shulman, 2023 – *LNL polycategories and doctrines of linear logic*

- Unbiased multicategories: very nice framework if we allow arbitrary families instead of sequences

  Pisani, 2025 – *Unbiased multicategory theory*

- Generalized multicategories: replacing lists of inputs with other structure.

  Symmetric, cartesian, skew multicategories

  Topological spaces, $\mathcal{U}(X) \nrightarrow X$

  Metric spaces

  Virtual double categories

  Leinster, 2004 – *Higher categories, higher operads*

  Cruttwell and Shulman, 2010 – *A unified framework for generalized multicategories*

Summary

Multicategories are an algebraic gadget providing an interface between logic and algebra.

- (planar) multicategories capture: context-free derivations, substructural logic
- symmetric multicategories: exchange in logic, commutative theories
- cartesian multicategories capture: usual logic, algebraic theories
- representability: cartesian products, tensor products, conjunction

Research in multicategory theory and its applications is ongoing!

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