Context-Free Languages of String Diagrams

Matt Earnshaw
Tallinn University of Technology

j.w.w. Mario Román (Oxford)

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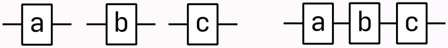
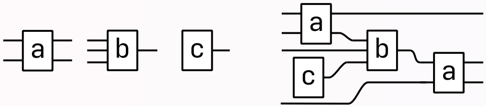
Introduction

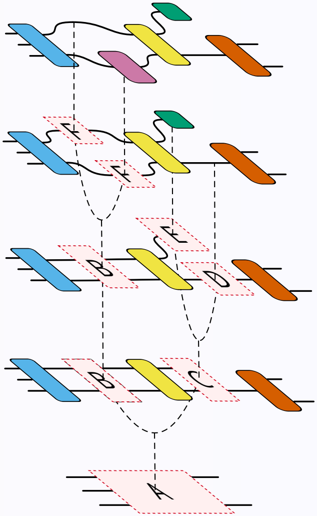
Formal languages of *words* live in monoids:



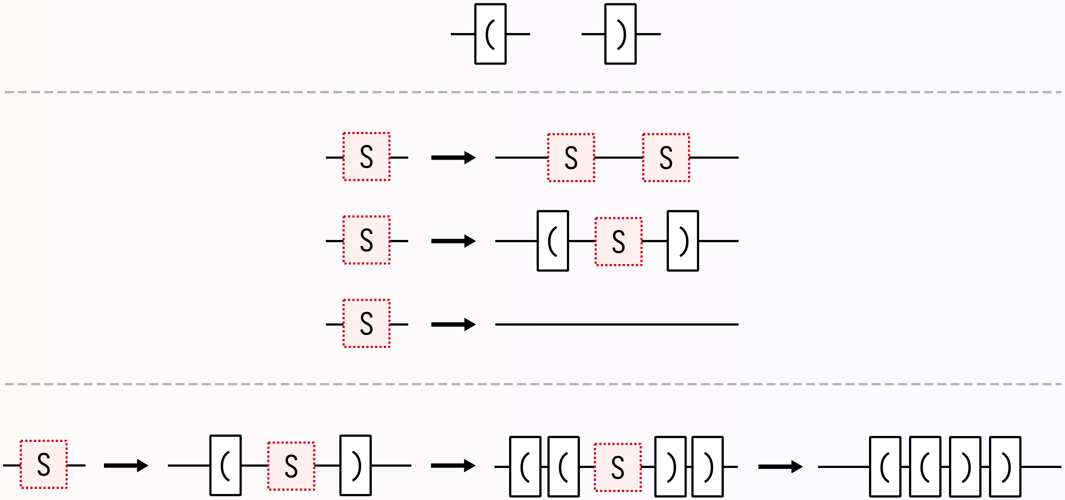
This talk is about languages that live in an algebraic gadget called *monoidal categories*:



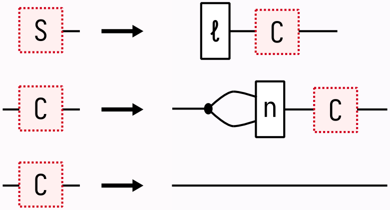
The resulting languages of string diagrams includes languages of words, trees, hypergraphs, and more, and involves some interesting maths.

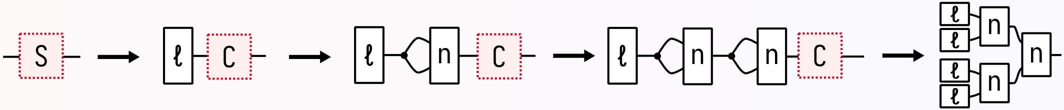


Example: context-free grammars

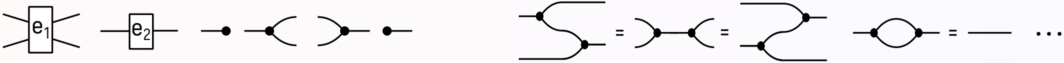


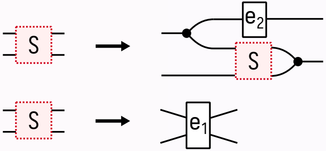
Example: context-free tree grammars

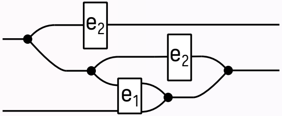




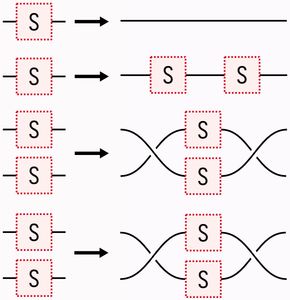
Example: context-free hypergraph grammars

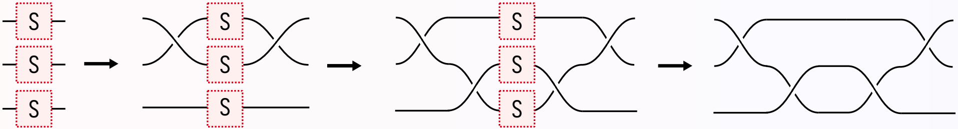




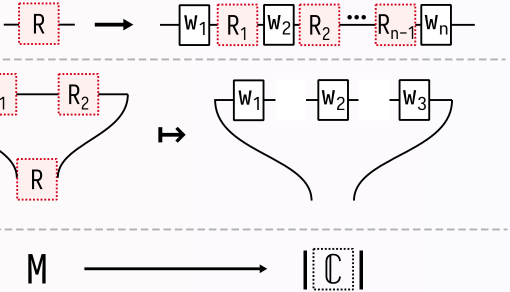


Example: context-free grammar of unbraids





Context-free languages á la Melliès-Zeilberger and Walters [MZ23, Wal89]



For $\mathbb{C}$ a category, $\boxed{\mathbb{C}}$ is a multicategory[1] with

- Objects: pairs of objects of $\mathbb{C}$, denoted $\genfrac{}{}{0pt}{}{A}{B}$
- $\boxed{\mathbb{C}}(;\genfrac{}{}{0pt}{}{X}{Y}) = \mathbb{C}(X;Y)$,
- $\boxed{\mathbb{C}}(\genfrac{}{}{0pt}{}{A_1}{B_1},\ldots,\genfrac{}{}{0pt}{}{A_n}{B_n};\genfrac{}{}{0pt}{}{X}{Y}) = \mathbb{C}(X;A_1)\times\prod_{i=1}^{n-1}\mathbb{C}(B_i;A_{i+1})\times\mathbb{C}(B_n;Y)$,
- composition, splicing into holes using composition in $\mathbb{C}$

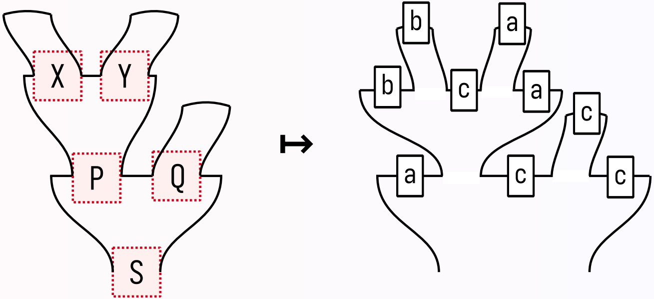
[1] Moreover, a *malleable* multicategory [ERH, Rom23]

Context-free languages á la Melliès-Zeilberger and Walters [MZ23, Wal89]

$$\mathtt{M} \xrightarrow{\ \Phi\ } |\mathbb{C}|$$

$$\mathtt{FM} \xrightarrow{\ \Phi^*\ } \mathbb{C}$$

Language of the grammar: $\Phi^*[FM(;S)] \subseteq \mathbb{C}(A;B)$



Context-free languages á la Melliès-Zeilberger and Walters [MZ23, Wal89]

$\mathbb{C}$: MonCat → MultiCat has a left adjoint given by *contours*.

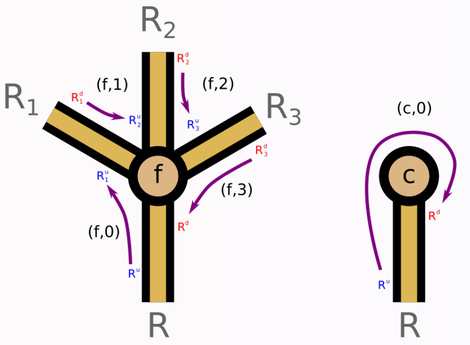
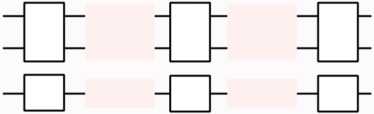


Figure: from Melliès and Zeilberger [MZ23].

What is an appropriate category of *monoidal* contexts (with a left adjoint)?

Categories of contexts I: monoidal multicategories of spliced arrows

For $(\mathbb{C}, \otimes, I)$ monoidal, the multicategory of Melliès and Zeilberger is monoidal:

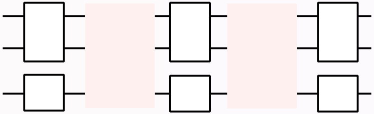


- Unit object $\genfrac{}{}{0pt}{}{I}{I}$
- $\otimes : \boxed{\mathbb{C}}_{\mathrm{obj}} \times \boxed{\mathbb{C}}_{\mathrm{obj}} \to \boxed{\mathbb{C}}_{\mathrm{obj}} : (\genfrac{}{}{0pt}{}{A}{B}, \genfrac{}{}{0pt}{}{C}{D}) \mapsto \genfrac{}{}{0pt}{}{A \otimes C}{B \otimes D}$,
- $\otimes_n : \boxed{\mathbb{C}}(\genfrac{}{}{0pt}{}{A_1}{B_1}, \ldots, \genfrac{}{}{0pt}{}{A_n}{B_n}; \genfrac{}{}{0pt}{}{X}{Y}) \times \boxed{\mathbb{C}}(\genfrac{}{}{0pt}{}{C_1}{D_1}, \ldots, \genfrac{}{}{0pt}{}{C_n}{D_n}; \genfrac{}{}{0pt}{}{U}{V}) \to \boxed{\mathbb{C}}(\genfrac{}{}{0pt}{}{A_1}{B_1} \otimes \genfrac{}{}{0pt}{}{C_1}{D_1}, \ldots, \genfrac{}{}{0pt}{}{A_n}{B_n} \otimes \genfrac{}{}{0pt}{}{C_n}{D_n}; \genfrac{}{}{0pt}{}{X}{Y} \otimes \genfrac{}{}{0pt}{}{U}{V})$: $((f_1, \ldots, f_{n+1}), (g_1, \ldots, g_{n+1})) \mapsto (f_1 \otimes g_1, \ldots, f_{n+1} \otimes g_{n+1})$,
- unit morphisms $i_n := (\mathrm{id}_I, \overset{n+1}{\ldots}, \mathrm{id}_I) \in \boxed{\mathbb{C}}(\genfrac{}{}{0pt}{}{I}{I}, \overset{n}{\ldots}, \genfrac{}{}{0pt}{}{I}{I}; \genfrac{}{}{0pt}{}{I}{I})$

This has a left adjoint, but the grammars are not expressive enough.

Categories of contexts I: monoidal multicategories of spliced arrows

For $(\mathbb{C}, \otimes, I)$ monoidal, the multicategory of Melliès and Zeilberger is monoidal:



- Unit object $\genfrac{}{}{0pt}{}{I}{I}$
- $\otimes : \mathbb{C}_{\text{obj}} \times \mathbb{C}_{\text{obj}} \to \mathbb{C}_{\text{obj}} : (\genfrac{}{}{0pt}{}{A}{B}, \genfrac{}{}{0pt}{}{C}{D}) \mapsto \genfrac{}{}{0pt}{}{A \otimes C}{B \otimes D},$
- $\otimes_n : \mathbb{C}(\genfrac{}{}{0pt}{}{A_1}{B_1}, \ldots, \genfrac{}{}{0pt}{}{A_n}{B_n}; \genfrac{}{}{0pt}{}{X}{Y}) \times \mathbb{C}(\genfrac{}{}{0pt}{}{C_1}{D_1}, \ldots, \genfrac{}{}{0pt}{}{C_n}{D_n}; \genfrac{}{}{0pt}{}{U}{V}) \to \mathbb{C}(\genfrac{}{}{0pt}{}{A_1}{B_1} \otimes \genfrac{}{}{0pt}{}{C_1}{D_1}, \ldots, \genfrac{}{}{0pt}{}{A_n}{B_n} \otimes \genfrac{}{}{0pt}{}{C_n}{D_n}; \genfrac{}{}{0pt}{}{X}{Y} \otimes \genfrac{}{}{0pt}{}{U}{V}) :$ $((f_1, \ldots, f_{n+1}), (g_1, \ldots, g_{n+1})) \mapsto (f_1 \otimes g_1, \ldots, f_{n+1} \otimes g_{n+1}),$
- unit morphisms $i_n := (\mathrm{id}_I, \overset{n+1}{\ldots}, \mathrm{id}_I) \in \mathbb{C}(\genfrac{}{}{0pt}{}{I}{I}, \overset{n}{\ldots}, \genfrac{}{}{0pt}{}{I}{I}; \genfrac{}{}{0pt}{}{I}{I})$

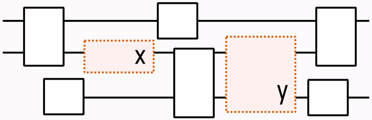
This has a left adjoint, but the grammars are not expressive enough.

Categories of contexts II: multicategories of contexts/wiring diagrams

$$\frac{f \in \mathbb{C}(A;B)}{\vdash f : \genfrac{}{}{0pt}{}{A}{B}} \qquad \frac{A, B \in \mathbb{C}_{\mathrm{obj}}}{\boxed{x} : \genfrac{}{}{0pt}{}{A}{B} \vdash \boxed{x} : \genfrac{}{}{0pt}{}{A}{B}} \qquad \frac{\Gamma_0 \vdash f_0 : \genfrac{}{}{0pt}{}{A_0}{A_1} \qquad \dots \qquad \Gamma_n \vdash f_n : \genfrac{}{}{0pt}{}{A_n}{A_{n+1}}}{\Gamma_0, \dots, \Gamma_n \vdash f_0; \dots ; f_n : \genfrac{}{}{0pt}{}{A_0}{A_{n+1}}}$$

$$\frac{\Gamma_0 \vdash f_0 : \genfrac{}{}{0pt}{}{A_0}{B_0} \qquad \dots \qquad \Gamma_n \vdash f_n : \genfrac{}{}{0pt}{}{A_n}{B_n}}{\Gamma_0, \dots, \Gamma_n \vdash f_0 \otimes \cdots \otimes f_n : \genfrac{}{}{0pt}{}{A_0 \otimes \cdots \otimes A_n}{B_0 \otimes \cdots \otimes B_n}}$$

... quotiented by associativity, unitality, interchange, and equations in $\mathbb{C}$.

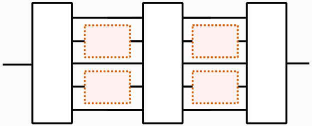


Problem: no left adjoint to $\boxed{}$: MonCat $\to$ Multicat

Categories of contexts III: “duomulticategories” of wiring diagrams

$$\frac{f \in \mathbb{C}(A; B)}{\vdash f : \genfrac{}{}{0pt}{}{A}{B}} \qquad \frac{A, B \in \mathbb{C}_{\mathrm{obj}}}{\boxed{x} : \genfrac{}{}{0pt}{}{A}{B} \vdash \boxed{x} : \genfrac{}{}{0pt}{}{A}{B}} \qquad \frac{\Gamma_0 \vdash f_0 : \genfrac{}{}{0pt}{}{A_0}{A_1} \qquad \dots \qquad \Gamma_n \vdash f_n : \genfrac{}{}{0pt}{}{A_n}{A_{n+1}}}{\Gamma_0 \triangleleft \cdots \triangleleft \Gamma_n \vdash f_0; \dots; f_n : \genfrac{}{}{0pt}{}{A_0}{A_{n+1}}}$$

$$\frac{\Gamma_0 \vdash f_0 : \genfrac{}{}{0pt}{}{A_0}{B_0} \qquad \dots \qquad \Gamma_n \vdash f_n : \genfrac{}{}{0pt}{}{A_n}{B_n}}{\Gamma_0 \otimes \cdots \otimes \Gamma_n \vdash f_0 \otimes \cdots \otimes f_n : \genfrac{}{}{0pt}{}{A_0 \otimes \cdots \otimes A_n}{B_0 \otimes \cdots \otimes B_n}}$$

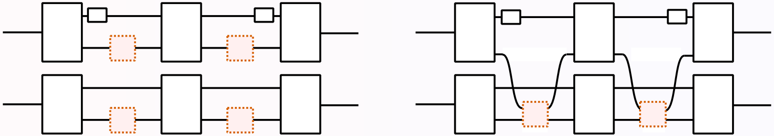


$$(A \otimes B) \triangleleft (C \otimes D); X$$

Arises from two adjunctions, but they don’t compose.

Categories of contexts IV: monoidal multicategories of combs

When $\mathbb{C}$ is symmetric, we have a middle ground.



Let $(\mathbb{C}, \otimes, I)$ be a symmetric monoidal category. Define:

$$\boxed{\mathbb{C}}\left(; {A \atop B}\right) := \mathbb{C}(A; B)$$

$$\boxed{\mathbb{C}}\left({C_1 \atop D_1}, \ldots, {C_n \atop D_n}; {A \atop B}\right) :=$$

$$\int^{X_1, \ldots, X_n} \mathbb{C}(A; X_1 \otimes C_1) \times \prod_{i=1}^{n-1} \mathbb{C}(X_i \otimes D_i; X_{i+1} \otimes C_{i+1}) \times \mathbb{C}(X_n \otimes D_n; B)$$

Not clear that left adjoint exists, but it does before quotienting, and this is enough!

Context-free monoidal grammars

Definition

A monoidal multigraph M is given by a set M_{obj} of objects, and for every pair of a list of lists $(X_1^1, \ldots, X_{n_1}^1)$, $\ldots, (X_1^k, \ldots, X_{n_k}^k)$, and a list $Y_1, \ldots, Y_n$ over M_{obj}, a set of morphisms

$$M((X_1^1 \otimes \ldots \otimes X_{n_1}^1), \ldots, (X_1^k \otimes \ldots \otimes X_{n_k}^k); Y_1 \otimes \ldots \otimes Y_n).$$

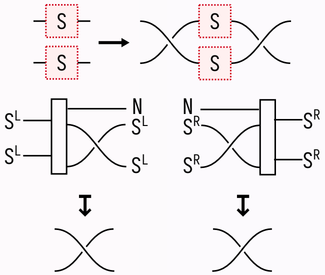
Definition

A *context-free monoidal grammar* over a symmetric monoidal category $\mathbb{C}$ is a morphism of monoidal multigraphs $M \to |\mathbb{C}|$ and a family of start symbols $S_{n,m} \in M_{\text{obj}}$.

A representation theorem for context-free languages of string diagrams

In previous work we introduced the class of *regular* languages of string diagrams [ESa, ESb]. Recognized by automata in which transitions take vectors of states to vectors of states.

$$FM \xrightarrow{\phi} \overline{\mathbb{C}} \to \mathbb{C} \qquad \mathcal{C}(\mathcal{F}M) \xrightarrow{\hat{\phi}} \mathbb{C}$$



Theorem

Every context-free monoidal language arises as the image of a regular monoidal language under a monoidal functor.

References

[ERH] M. Earnshaw, M. Román, and J. Hefford, *The Produoidal Algebra of Process Decomposition*, 32nd EACSL Annual Conference on Computer Science Logic (CSL 2024).

[ESa] M. Earnshaw and P. Sobociński, *Regular monoidal languages*, MFCS 2022, LIPIcs, pp. 44:1–44:14.

[ESb] ———, *String Diagrammatic Trace Theory*, MFCS 2023, LIPIcs, pp. 43:1–43:15.

[MZ23] Paul-André Melliès and Noam Zeilberger, *The categorical contours of the Chomsky-Schützenberger representation theorem*, December 2023.

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[Wal89] R.F.C. Walters, *A note on context-free languages*, Journal of Pure and Applied Algebra **62** (1989), no. 2, 199–203.